ELECTRONS WITH SELF-FIELD AS SOLUTIONS TO NONLINEAR PDE

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ABSTRACT. The Maxwell-Dirac equations give a model of an electron in an electromagnetic (e-m) field, in which neither the Dirac or the e-m fields are quantized. The two equations are coupled *via* the Dirac current which acts as a source in the Maxwell equation, resulting in a nonlinear system of partial differential equations (PDE's). In this way the *self-field* of the electron is included.

We review our results to date and give the four real *consistency conditions* (one of which is conservation of charge) which apply to the components of the wavefunction and its first derivatives. These must be met by any solutions to the Dirac equation. These conditions prove to be invaluable in the analysis of the nonlinear system, and generalizable to higher dimensional supersymmetric matter.

In earlier papers, we have shown analytically that in an isolated stationary system, the surrounding electon field *must* be equal and opposite to the central (external) field. The nonlinearity forces electric neutrality, at least in the static case. We illustrate these properties with a numerical family of orbits which occur in the (static) spherical and cylindrical ODE cases. These solutions are highly localized and die off exponentially with increasing distance from the central charge.

1. The Maxwell-Dirac Equations and QED

The coupled Maxwell-Dirac equations can be written as follows:

$$\begin{aligned} \gamma^\alpha(\partial_\alpha - ieA_\alpha)\psi + im\psi &= 0 \quad \text{where} \quad \alpha = 0, \dots, 3 \\ F_{\alpha\beta} &= A_{\beta,\alpha} - A_{\alpha,\beta} \\ \partial^\alpha F_{\alpha\beta} &= -4\pi e j_\beta \\ \text{where} \quad j_\alpha &= \overline{\psi}\gamma_\alpha\psi. \end{aligned} \tag{1}$$

Note that $\psi \in C^4$ is the Dirac wave-function or *4-spinor* and $\overline{\psi}$ the Dirac conjugate. These are acted upon by γ^α which are the usual *gamma matrices* (representations of a Clifford algebra), and A_α is the 4-potential.

These equations model an electron in an electromagnetic field. The two equations are coupled *via* the *Dirac current* j_α i.e. we include the nonlinearity of the *self-field*, see Equation (1). These equations form the foundation of quantum electrodynamics (QED), the theory of electrons interacting with fields. QED is one of the most successful physical theories, explaining the Lamb shift and the anomalous magnetic moment of the electron. The calculations of QED are acheived by quantizing the fields and using perturbation theory. In so doing, well-known mathematical problems occur, which have yet to be resolved at the fundamental level.

It is possible that the full nonlinearized equations must be analysed rigorously before we can hope to resolve these deep problems. For example, in [23], Lieb and Loss observe that the stability of matter requires that the electron be defined with a Dirac operator *with* the magnetic vector potential $A(x)$, instead of the free Dirac operator (without $A(x)$). That perturbation theory must start from the *dressed* electrons (including their own *self-field*) might be "fundamentally important in a non-perturbative QED".

This view is shared by many analysts working on this problem including Flato, Simon and Taflin who established global existence for the M-D equations as recently as 1997 [18], following many years of sustained interest [21] [9] [20] [19] in the problem. In [18], Flato *et al* showed that the nonlinear representation is integrable to a global nonlinear representation of the Poincaré group on a differential manifold, $\mathcal{U}_\infty$ of small initial conditions. This established the existence of global solutions for initial data in $\mathcal{U}_\infty$ at $t = 0$. They go on to show that the asymptotic representations are also nonlinear and draw conclusions for the infrared tail of the electron. Their results show that "in the classical case (also), one obtains infrared divergencies, *if one requires free asymptotic fields* as it is needed in QED". In other words, the fields must remain coupled *via* the self-field if we are to resolve the infrared problem.

In the case where we assume that the system is *static* and/or *stationary* (see Section 2), we can make some simplifications. The system is elliptic in the stationary case (no time dependence). Esteban, Georgiev and Séré showed existence of *soliton-like* solutions (that is, solutions which are spatially localized) in this case [16]. Furthermore, the wave-function, together with all its derivatives decreases exponentially at infinity.

2. Some simplified versions of the problem

The full 4-dimensional nonlinear problem is somewhat intractable – as stated in Section 1, global existence has only been established recently [18] (and references therein). If we want to get some idea of the types of behaviour we might expect in the 4-dimensional problem, we might begin by looking at various subcases.

Some simplified versions of the problem are as follows:

1. The static case in which we assume that there exists a Lorentz frame in which there is no current "flow" i.e. $j_\alpha = \delta^0_\alpha j_0$ [29] [4],
2. The stationary case in which we assume $\psi(x_0, \boldsymbol{x}) = e^{i\omega x_0}\, \phi(\boldsymbol{x})$ [16],
3. The static spherically symmetric case [29],
4. The static cylindrically symmetric case [4] [8],
5. The static case with z dependence only [6],
6. The $1+1$ case which was solved exactly for a massless electron by Schwinger in [31],
7. The static axi-symmetric case,
8. The circular current case in which we assume (in spherical coordinates) that $j_\alpha = (j_0, 0, j_\phi, 0)$.

It appears that the static and stationary assumptions are rather strong, since the resulting system becomes elliptic rather than hyperbolic. In [5] and [30] we proved electric neutrality in the static case (for an isolated system). While this is interesting in that it implies that solutions of this type must consist of an inner charge, say, surrounded by an equal and oppositely charged electron field i.e. the solutions must be *atom-like*, it raises the problem of finding a solution that represents a single charged particle. We need to find a weaker ansatz, perhaps the circular current assumption, with which we have sufficient simplification without losing the important properties of the 4-dimensional system. It is possible of course, that electric neutrality could be shown to be fully general, which would supporting the conjecture that the total charge of the universe is zero. The larger (quantum cosmological) problem here is the Einstein-Maxwell-Dirac problem, and related problems such as Einstein-Dirac [17] and Einstein-Yang-Mills [3].

Following the methods of the analysts working in relativity theory, our aim in [29] [4] etc was to enumerate the subcases, starting with all possible ODE cases (see Section 4), and then progressing to (static) axi-symmetric and other two-dimensional cases. Meanwhile, the three dimensional static case proved to be somewhat tractable.

3. Using the Clifford Algebra to derive some constraints

In this section, we make use of the properties of the Clifford algebra (which the γ^α represent as 4×4 matrices), to solve the Dirac equation for the potential and to show that there are some useful consisitency conditions upon the wavefunction and its first derivatives. We want to write the potential in terms of the wavefunction, so that we can substitute it into the Maxwell equation.

A complex Clifford algebra $\mathcal{A}(n)$ consists of all possible products of the n basis vectors (of an n-dimensional vector space), e^i, which obey the following:

$$(e_i)^2 = -1, \text{if } i \in \{0, \dots, n-1\}, \\ e_i e_j + e_j e_i = 0, \text{if } i \neq j. \tag{2}$$

Using the isomorphism [11]

$$\mathcal{A}(n+2) \cong \mathcal{A}(n) \otimes \mathcal{A}(2) \tag{3}$$

we can construct the representation of $\mathcal{A}(4)$, the *gamma matrices*, from the Pauli matrices which represent $\mathcal{A}(2)$. For the construction of higher dimensional Clifford algebras from lower dimensions see for example [14]. One possible representation of $\mathcal{A}(4)$ is:

$$\gamma^0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \gamma^1 = i \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^2 = i \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \gamma^3 = i \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{4}$$

We note that in this representation, γ^0 is anti-symmetric (a-s), and $\gamma^i, i = 1, 2, 3$ are symmetric.

Whatever representation we use, we can always invert the Dirac equation to express the spatial components of the potential, A^i, $i = 1, 2, 3$ in terms of A^0, the wavefunction, ψ and its first derivatives. Multiplying (1) on the left $\psi^t \gamma^i$ (where ψ^t is the transpose of ψ) we have

$$\psi^t \gamma^i \gamma^\alpha A_\alpha \psi = \frac{1}{ie} \psi^t \gamma^i \left(\gamma^\alpha \partial_\alpha + im\right) \psi. \tag{5}$$

But the product $\gamma^i \gamma^j$, $i \neq j$ is (a-s), $\gamma^i \gamma^i = -1$, and $\gamma^i \gamma^0$ is symmetric, so that, using the argument following (7) we can write:

$$\psi^t \psi A_i = \psi^t \gamma^0 \gamma^i A_0 \psi + \frac{1}{ie} \psi^t \gamma^i \left(\gamma^\alpha \partial_\alpha + im\right) \psi. \tag{6}$$

In a similar way (by multiplying the Dirac equation by $\overline{\psi} \gamma^0 \gamma^1 \gamma^2 \gamma^3$ on the left, and subtracting the conjugate equation premultiplied by

$\psi\gamma^0\gamma^1\gamma^2\gamma^3$) we can solve for A_0. There is a condition to be met here see [29] i.e. that j^α is not *a null vector* which has the physical interpretation that the electron is not travelling faster than the speed of light.

In 1957, one of Dirac's students, Eliezer solved for the potential in this way [15]. He went on to show that when we solve for the potential, we must also adopt a consistency condition which applies to ψ. In that paper there was a contribution by Dirac who streamlined some of Eliezer's calculations. Although they used a different representation of the algebra, the argument was essentially the following.

If , is an antisymmetric (a-s) (4×4) matrix then

$$\psi^t , \psi = 0, \text{since this quantity is an a-s scalar.} \tag{7}$$

If we can find a , such that , γ^α are all a-s, then

$$\psi^t , \left(\gamma^\alpha A_\alpha\right)\psi = 0,$$

since A_α is a scalar, so that if ψ satisfies (1) then

$$\psi^t , \left(\gamma^\alpha \partial_\alpha + im\right)\psi = 0, \tag{8}$$

which gives us a consistency condition on ψ and its first derivatives.

The same is true of the complex Dirac equation and we have

$$\overline{\psi}^t , \left(\gamma^\alpha \partial_\alpha - im\right)\overline{\psi} = 0. \tag{9}$$

We can extend Dirac's argument one step further by premultiplying the Dirac equation by $\overline{\psi}$ and the complex equation by ψ and noting that since , γ^α is a-s,

$$\left(\overline{\psi}^t , \gamma^\alpha A_\alpha \psi\right)^t = -\psi^t , \gamma^\alpha A_\alpha \overline{\psi}.$$

This gives us another condition on ψ namely

$$\left(\overline{\psi}^t , \left(\gamma^\alpha \partial_\alpha + im\right)\psi\right)^t + \psi^t , \left(\gamma^\alpha \partial_\alpha - im\right)\overline{\psi} = 0. \tag{10}$$

There is only one possible element of the Clifford algebra which when premultiplying all of the γ^α yields an a-s matrix. In the representationused here, this is the product

$$, = \gamma^1\gamma^2\gamma^3.$$

Using this in (8) (9) and (10), we have two complex conditions (or four real conditions) on the components of ψ.

As is well-known (see [11] for example), even dimensional complex Clifford algebras are simple, that is, they can *not* be decomposed into the direct sum of two nontrivial subspaces which obey closure under algebraic multiplication. If we want to decompose the 4-dimensional into a 2-dimensional algebra and use 2-spinors, then when must accept the additional structure in which the two 2-spinor spaces are conjugate dual spaces. See for example [28] in which the 2-spinor formalism is given in terms of Infield-van der Waerden symbols. An argument, equivalent to Dirac/Eliezer's but giving all of the consistency conditions, was given by Radford using the 2-spinor formalism in 1996 [29], and subsequently in [4], although Radford referred to them as "reality conditions", in keeping with the conventions in [28].

We can think of the Dirac equation and its conjugate equations as eight equations in four unknowns (the four real scalars, A_α). If we solve for these A_α then we must have four additional (real) constraints, which correspond to the two complex consistency conditions, confirming that there are no further constraints upon the system. In [7] we outline these conditions and then go on to generalize to higher dimensional cases. Allowing these higher Clifford algebras enables us to pursue the same arguments when applied to supersymmetric matter [7]. See [22] and references therein.

4. ODE solutions

Within the static system there are three interesting ODE cases, spherically symmetric, cylindrically symmetric and dependence on z only. The spherical and cylindrical cases were examined extensively in [29] [4]. The case where dependence is on z only, is similar in some respects [5]. We first apply our *consistency conditions* to the electromagnetic potential A^α which has been expressed in terms of Dirac spinors and their first derivatives (by solving the Dirac equations for the potential as outlined in Section 3). These reality conditions allow us some simpler expressions which are then inserted into the Maxwell equation, resulting in fourth order ODE's. We will also note here that in the $1+1$ case [12] [13] [6] the system was also reduced to fourth order ODE's, which in some cases are solved explicitly [12] [13], whilst in others we are currently developing more numerical results [6].

When we assume that the Dirac current is static, we lose three (real) degrees of freedom in ψ, since three components of j^α are set to zero. See [29] and [4] in which we also fix the gauge by choosing

$$\begin{aligned} \psi^0 &= -Ye^{\frac{i}{2}(\chi-\eta)} & \psi^2 &= -Ye^{-\frac{i}{2}(\chi+\eta)} \\ \psi^1 &= Xe^{\frac{i}{2}(\chi+\eta)} & \psi^3 &= Xe^{-\frac{i}{2}(\chi-\eta)}\,, \end{aligned} \tag{11}$$

where η, χ, X and Y are real functions. These expressions were substituted into the potential (which was solved for in terms of Dirac spinors and their first derivatives), yielding:

$$A^0 = \cos\chi + \frac{(X^2-Y^2)}{(X^2+Y^2)}\frac{\partial\eta}{\partial t} + \frac{(\boldsymbol{\nabla}\chi).\boldsymbol{l}}{(X^2+Y^2)} \tag{12}$$

$$\boldsymbol{A} = \frac{1}{(X^2+Y^2)}\left[\frac{\partial\chi}{\partial t}\boldsymbol{l} + (X^2-Y^2)\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}\times\boldsymbol{l}\right]. \tag{13}$$

We now use the consistency conditions, one of which is conservation of charge, which is obeyed automatically as stated in [29]. The other three conditions, in the variables required for the static case, are given below.

$$\frac{\partial}{\partial t}(X^2+Y^2) = 0 \tag{14}$$

$$\boldsymbol{\nabla}.\boldsymbol{l} = -(X^2+Y^2)\sin\chi \tag{15}$$

$$\frac{\partial\boldsymbol{l}}{\partial t} + (\boldsymbol{\nabla}\chi)\times\boldsymbol{l} = \mathbf{0}. \tag{16}$$

where $\boldsymbol{l} = (2XY\cos\eta, 2XY\sin\eta, X^2-Y^2)$.

The Maxwell equations act upon $\boldsymbol{A}$, as defined above, and the current vector becomes $j^\alpha = (2(X^2+Y^2), 0, 0, 0)$.

We showed in [5] that the static equations, in the gauge given by Eq. 11, are stationary if and only if $\frac{\partial\eta}{\partial t} = 0$ and $\frac{\partial X}{\partial t} = 0$ (or $\frac{\partial Y}{\partial t} = 0$). In the stationary case, $\frac{\partial\chi}{\partial t} = 0$ and $\frac{\partial\boldsymbol{l}}{\partial t} = 0$. Now in the stationary case, the third reality condition Eq. 16 tells us that $\boldsymbol{\nabla}\chi$ is proportional to $\boldsymbol{l}$ and we choose the function α such that $\boldsymbol{l} = \frac{\alpha}{r\sin\theta}\ \boldsymbol{\nabla}\chi$. Substituting this into the expression for the potential Eq. 12 and noting that $X^2+Y^2 = |\boldsymbol{l}|$ then

$$A^0 = \cos\chi \pm \frac{\boldsymbol{\nabla}\chi.\boldsymbol{l}}{(X^2+Y^2)} = \cos\chi \pm \sqrt{\chi_r^2 + \frac{\chi_\theta^2}{r^2} + \frac{\chi_\phi^2}{r^2\sin^2\theta}}. \tag{17}$$

From here, it is a straightforward calculation to apply symmetry arguments and calculate the resulting ODE's [29] [4] [5].

Then, in dimensionless variables [29] [4], the equations reduce to:

$$\begin{aligned} \frac{d\chi}{dx} &= A - \cos\chi \\ \frac{dF}{dx} &= Z \\ \frac{dA}{dx} &= \frac{F}{f(x)} \\ \frac{dZ}{dx} &= -Z\sin\chi, \end{aligned} \tag{18}$$

where

$$f(x) = \begin{cases} x^2 & \text{in the spherical case} \\ x & \text{in the cylindrical case} \\ 1 & \text{in the } z \text{ dependent case} \end{cases}$$

and

$$x = \begin{cases} r & \text{in the spherical case} \\ \rho & \text{in the cylindrical case} \\ z & \text{in the } z \text{ dependent case,} \end{cases}$$

r being the distance from the origin and ρ the distance from the central axis.

It is easily shown that these four first order equations can be written as the fourth order,

$$\frac{d^2}{dx^2}\left(f(x)\left(\frac{d^2\chi}{dx^2} - \sin\chi\frac{d\chi}{dx}\right)\right) + \frac{d}{dx}\left(f(x)\left(\frac{d^2\chi}{dx^2} - \sin\chi\frac{d\chi}{dx}\right)\right) \quad \sin\chi = 0. \tag{19}$$

In the cylindrical case, $Z(\rho)$ is the charge per unit ring radius ρ, $F(\rho)$ is the charge *within* a radius ρ, and A is the scalar potential, A^0. See [4]. Similar physical quantities are represented in the spherical and z cases. Most importantly, we are looking for solutions whose charge density $Z(\rho)$ decreases rapidly towards infinity, so that we can find solutions which are localized or *particle-like.* Our zero total charge result [5] tells us to expect that $F \to 0$ as $\rho \to \infty$ and likewise, $A \to const$ as $\rho \to \infty$.

As pointed out by Chris Cosgrove (private communication), equation (19) has non-integer resonance numbers [1] [2] and we do not expect to find an integrable system (in the soliton sense) here. Instead, we show that there are a family of *orbits* (in the sense of [10]) all of which approach the trivial (constant) solution at infinity. As an example, we look at the cylindrical case, noting that similar results hold in the spherical and z case. The numerical orbits in the Figure 1 complete the results in [4] in which a single member (analytic in $\frac{1}{\rho}$) of these families was shown to exist and calculated numerically.

5. Vacuum Maxwell singularities near the origin

In [4] it was shown that in the cylindrical case

Lemma 1. *Suppose (χ, F, A, Z) is a solution to Equation(18) on $I = (0, \rho_1)$, for some $\rho_1, 0 < \rho_1 < 1$. Suppose also that $Z \geq 0$ is continuous and bounded on I. Then,*

(i) *F is C^1 on I and has a well defined, finite limit as $\rho \to 0$. Z has a well-defined limit as $\rho \to 0$.*
(ii) *if $F(0) \neq 0$ then A is unbounded as $\rho \to 0$. In particular, $A = \Omega(\rho)\ln(\rho)$, where Ω is C^2 and bounded on I, $\Omega \to F(0)$ as $\rho \to 0$. Also, χ is bounded as $\rho \to 0$.*

Lemma 2. *Suppose (χ, F, A, Z) is a solution to Equation(18) on $\rho \in (0, \infty)$. Suppose also that $Z \geq 0$ with F continous and bounded on the interval. Then*

(i) *If $F(\rho_1) \geq 0$ for some $\rho_1 \in [0, \infty)$, then $\chi \to \infty$, $A \to \infty$, and $F \to \infty$ as $\rho \to \infty$.*
(ii) *If $F < 0$ on $(0, \infty)$ then $F \to 0$ as $\rho \to \infty$. In addition, if A and Z have well-defined limits as $\rho \to \infty$ then $Z \to 0$ and $A \to A_\infty$ as $\rho \to \infty$, with $-1 \leq A_\infty \leq 1$.*

Similar results were established for the spherical case in [29]. In [29][4], Radford and the current author have shown that the behaviour of solutions near the origin resembles the vacuum solutions of the Maxwell equations. Given that the Maxwell equation can be written as the square of a Dirac operator (see for example [26]), it should be possible to formulate these results, together with exponential decrease in terms of the properties of *k-monogenic* functions — those functions which are solutions to

$$D_k = D + k\epsilon_0 = 0,$$

where ϵ_0 is a basis vector (corresponding to the time coordinate) of a complex Clifford algebra. The fundamental solution of the k-Dirac operator has the same singularity at the origin and decreases exponentially as $x \to \infty$. In this case, the operator has coefficients in Λ^1, the subspace of vectors γ^α. We are currently working on a clarification of this point.

6. Numerical Solutions

In [29][4] [8] we showed examples of numerical solutions exhibiting the characteristics referred to in Section 5. Earlier attempts at finding numerical solutions[32] [25] were marred by a "simplifying assumption" which proved to be invalid – solutions did not exist in that case. See [29].

In [8] we noted that there are families of orbits parameterized by the constant $c|x|$ where we assume that solutions are of the form $e^{-c|x|}g(|x|)$, with $g(|x|)$ an analytic function. By varying the value of c, the boundary conditions are perturbed to neighboring solutions (orbits), all decreasing exponentially at infinity. See Figure 1. All values of c yield solutions which satisfy the two Lemmas, and the zero total charge result. That is, all solutions surround a central wire along the axis of symmetry.

Similar numerical solutions can be found in the spherical case [6]. The spherical solutions surround a central Coulomb field, but the static condition forces a monopole at the origin [29] [5] [30], that is, an unbounded A^ϕ component. Further ODE solutions occur in the z case [5] and $1+1$ case, neither of which have been fully examined in the previous literature [12] [13]. These results will be forthcoming also in [6].

Similar spherically-symmetric solutions were found in [27]. This time the Schrödinger-Newton equations provided an identical coupling as in the static Maxwell-Dirac case, which is essentially an elliptic system. More work must be done to investigate the stability of both of these systems. In [27] implications have been developed in the context of quantum gravity, but work in this area is far from complete. We are currently considering the problem in this context. The solutions found in [27] blow-up at larger distances from the origin. However, in the Maxwell-Dirac case, this behaviour appeared only as a numerical anomaly. When the total charge became slightly positive (due to the step-size of the numerical solution), we entered a regime described in Lemma 2 of [4] in which all solutions become unbounded as $\rho \to \infty$. These solutions were illustrated numerically in [4], but discarded as being of less interest than the bounded solutions.

The solution shown in Figure 1 was calculated using the MATLAB ODE solver ODE113. The relative error tolerance was set at $1e-4$ and the absolute error tolerance at $1e-8$. The same behaviour was observed when the tolerances were decreased to $1e-6$ and $1e-12$ or increased to $1e-3$ and $1e-6$. All solutions remained stable whether calculated as a function of increasing or decreasing radius (i.e. shooting away from or towards the central axis).

The two dimensional cases (static axi-symmetric, circular current axi-symmetric, the massive $1+1$ case) still require reliable numerical results. Those available to date [32] [25] have been flawed by the imposition of "approximations" which were shown in [29] to be possible only in trivial cases (in which the equations are no longer coupled).

7. Vanishing Total Charge

In the solutions in Figure 1 the variable $F(\rho)$ representing the total charge within a ring, radius ρ, tends towards zero at ∞. Lemma 1 also states that the potential must tend towards a solution which is

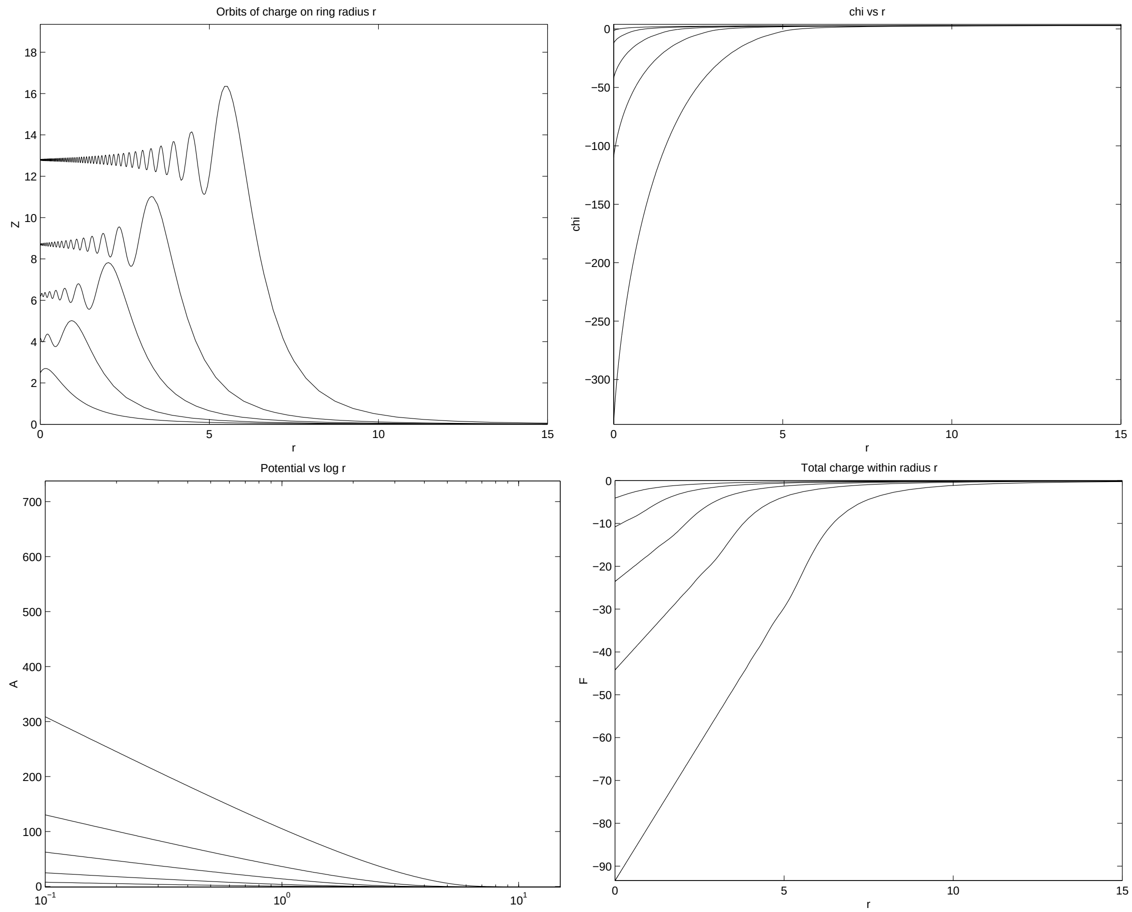

FIGURE 1. A family of orbits exhibiting the properties of Lemma 1 and Lemma2.

logarithmic in ρ, near the central axis. This corresponds to a central charged wire (along the z axis) which is a solution to the *vacuum* Maxwell equations. As such, we can think of this part of the solution as representing an *external field*. (The scalar potential could be separated at this point into $A_{external} + A_{fermion+interaction}$, since the $A_{external}$, as the homogenous solution, contibutes nothing to the coupling between the Maxwell and Dirac equations.) Physically, this means that the (inner) external field must be surrounded by an equal and oppositely charged field. The total electric charge of the system

$$\lim_{\rho\to\infty} \frac{1}{4\pi} \int_{S_\rho} \left(\boldsymbol{\nabla} A^0\right) \boldsymbol{.} d\boldsymbol{S}, \quad \text{with} \quad S_\rho \quad \text{the unit ring of radius } \rho,$$

must vanish.

In [5], we were able to show that this is the case for all stationary solutions, given that they are in a "reasonable" function space. If we define an **isolated system** as one for which all sources are contained in some ball B_k $(k < \infty)$ and for which the fields die off as $|x| = r \to \infty$, then we showed in [5] and [30] that an isolated, stationary, static Maxwell-Dirac system is electrically neutral. A similar result, for the total electric charge per unit length in the z direction (or *unit ring*), holds in the cylindrical case. This shows **mathematically**, that a (stationary) solution must be atom-like (in the sense that any central charge must be surrounded by an equal and opposite charge. In addition it shows that there **cannot** be a stationary solution representing an isolated electron (or even one that rotates around the axis with constant velocity).

Note also, that the signs of the charges can be reversed, giving us negatively charged singularites and positively charged "electron" fields. All results remain unchanged under such a reversal of sign. The total charge must vanish overall. A more difficult problem presents itself if we are to allow for two fermion fields of opposite charge, for example, the solution corresponding to positronium. We are currently considering ways in which we can have have solutions of opposite charge interacting together.

8. Discussion and Conclusions

Until the mathematical problems of QED have been resolved, all methods of addressing the equations governing the interaction between electrons and fields are significant. A non-perturbative QED must consist of classical and/or semi-classical arguments which aim to justify the Feynman integral method, and the quantized field approach.

For the coupled M-D system, we have existence (for small initial data), existence for the stationary case, and descriptions of various subcases. But analysis of the system is far from complete. We have indicated, within this review paper, the directions we are currently pursuing.

References

[1] M.J.Ablowitz,A.Ramani,H.Segur *A connection between nonlinear evolution equations and ordinary differential equations of P-type.I* J.Math.Phys. **21** (4), 715-721 (1980)

[2] M.J.Ablowitz,A.Ramani,H.Segur *A connection between nonlinear evolution equations and ordinary differential equations of P-type.II* J.Math.Phys. **21** (5), 1006-1018 (1980)

[3] R.Bartnik, J McKinnon, *Particlelike solutions of the Einstein-Yang-Mills equations* Phys. Rev. Lett., **61** 141-144, 1988.
[4] H.S.Booth, C.J.Radford, *The Dirac-Maxwell equations with cylindrical symmetry* J. Math. Phys. **38** (3), 1257-1268 (1997).
[5] H.S.Booth, *The Static Maxwell-Dirac Equations* PhD Thesis, University of New England (1998).
[6] H.S.Booth *Various ODE Solutions to the static and* $1+1$ *Maxwell-Dirac Equations* (in preparation).
[7] H.S.Booth,P.Jarvis,G.Legg, *Algebraic solution for the vector potential in the Dirac equation* (in preparation).
[8] H.S.Booth, *Nonlinear electron solutions and their characteristics at infinity* The ANZIAM Journal (formerly J.Aust.M.S (B)) (to appear).
[9] J.Chadam *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac system in one space dimension* J.Funct.Anal. **13**, 173-184 (1973).
[10] K.Cieliebak and E.Séré, *Pseudoholomorphic curves and multiplicity of homoclinic orbits*, Duke Mathematical Journal **77**, (2) 483-518 (1995).
[11] R.Coquereaux, *Modulo 8 periodicity of real Clifford algebras and particle physics*, Phys Lett **115B** (5), 389-395 (1982).
[12] A.Das, D.Kay, *A class of exact plane wave solutions of the Maxwell-Dirac equations*, J.Maths.Phys. **30** (10) (1989) 2280-2284.
[13] A.Das, *General Solutions of Maxwell-Dirac equations in 1+1 dimensional space-time and a spatially confined solution*, J.Maths.Phys. **34** (9) (1993) 3986-3999.
[14] R.Delanghe,F.Sommen,V.Souček, *Clifford Algebra and Spinor-Valued Functions*, KLUWER (1992).
[15] C.J. Eliezer, *A Consistency Condition for Electron Wave Functions*, Camb.Philos. **54**, 2 (1957).
[16] M.Esteban, V.Georgiev, E. Séré, *Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations*, Calc. Var. **4**, 265-281, (1996).
[17] Felix Finster, Niky Kamran, Joel Smoller, Shing-Tung Yau, *Nonexistence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry* CPAM **53**,7, 902-929 (2000).
[18] M. Flato, J.C.H. Simon, E Taflin, *Asymptotic Completeness, Global Existence and the Infrared Problem for the Maxwell-Dirac Equations* Memoirs of the American Mathematical Society, (May 1997).
[19] V.Georgiev, *Small amplitude solutions of the Maxwell-Dirac equations* Indiana Univ. Math. J. **40** (3), 845-883 (1991).
[20] R.T.Glassey, W.A.Strauss *Conservation Laws for the Maxwell-Dirac and Klein-Gordon-Dirac equations*, J.Math.Phys., **20**, (1979).
[21] L.Gross *The Cauchy problem for the coupled Maxwell and Dirac equations* Comm. Pure Appl. Math. **19** 1-5 (1966).
[22] P.S.Howe, G.Sierra, P.K.Townsend *Supersymmetry in six dimensions* Nuclear Phys **B221** 331-348 (1983).
[23] E.H. Lieb and M. Loss, *Self-Energy of Electrons in Non-perturbative QED* , in *Differential Equations and Mathematical Physics, University of Alabama, Birmingham, 1999*, R. Weikard and G. Weinstein, eds. 255-269 Internat. Press (1999). arXiv math-ph/9908020, mp_arc 99-305.

[24] E.H. Lieb and M. Loss, *Remarks about the ultraviolet problem in QED*, (in preparation).

[25] A.G. Lisi, *A solitary wave solution of the Maxwell-Dirac equations*, J.Phys.A **28** No.18, 5385-5392 (1995).

[26] A.McIntosh, *Clifford algebras, Fourier theory, singular integrals, and harmonic functions on Lipschitz domains* in: *Clifford Algebras in Analysis and Related Topics* (John Ryan ed), 33-88 CRC Press Boca Raton, FL, 1996.

[27] I.M.Moroz,R.Penrose,P.Tod, *Spherically-symmetric solutions of the Schrödinger-Newton equations*, Class. Quantum Grav. **15** 2733-2742 (1998).

[28] R.Penrose,W.Rindler, *Spinors and Space-Time I (Two Spinor Calculus and Relativistic Fields)*, CAMBRIDGE UNIVERSITY PRESS, (1984).

[29] C.J. Radford, *Localised Solutions of the Dirac-Maxwell Equations*, J. Math. Phys. **37** (9), 4418-4433 (1996).

[30] C.J.Radford, H.S.Booth *Magnetic Monopoles, electric neutrality and the static Maxwell-Dirac equations* J.Phys.A:Math.Gen. **32**, 5807-5822 (1999).

[31] J.Schwinger *Gauge Invariance and Mass. II* Phys. Rev. **128** (5), 2425-2429 (1962).

[32] M. Wakano, *Intensely Localized Solutions of the Dirac-Maxwell Field Equations*, Prog.T.Phys. **35**, 1117-1141, (1966).

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