SOME REMARKS ON OSCILLATORY INTEGRALS

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1. Introduction

The purpose of this note is to describe some results about oscillatory integral operators. Specifically we are interested in bounds in Lebesgue spaces of operators given by

$$T_\lambda f(x) = \int_{\mathbf{R}^k} e^{i\lambda\varphi(x,\xi)}\ f(\xi)\ d\xi,$$

with $\varphi(x,\xi)$ a real-valued smooth function on $\mathbf{R}^n \times \mathbf{R}^k, k \leq n$. Obviously T_λ is bounded as maps from L^q_{comp} to L^p_{loc}. What is of interest here is the dependence of the norm for increasing λ. This will of course depend on the conditions we put on the phase function φ. To guarantee that φ lives on an open subset of $\mathbf{R}^n \times \mathbf{R}^k$ it is natural to start with the condition

$$\text{rank}\ \ d_\xi d_x \varphi = k, \quad x \in \mathbf{R}^n \text{ and } \xi \in \mathbf{R}^k. \tag{1}$$

We will assume this condition throughout this note. For work related to weaker assumptions see, e.g. [21] and [18]. One of the questions we will ask is: What is the optimal (q,p)-range for which the operator T_λ has norm of order $\lambda^{-n/p}$? In particular we would like to understand how this range will depend on k.

To put things in perspective let us begin by describing what is known for the case $k = n$: A model phase function here is $\varphi(x,\xi) = x \cdot \xi$, for $x, \xi \in \mathbf{R}^n$. Then T_λ is a localized version of the Fourier transform and the (L^q_{comp}, L^p_{loc})-boundedness properties are covered by the Hausdorff-Young inequality. For general phase functions satisfying (1) the L^2-theory of Fourier integral operators gives

$$|||T_\lambda|||_{L^q_{comp}\to L^p_{loc}} \leq\ C\ \lambda^{-n/p},$$

with $p = q' \geq 2$ the dual exponent of q, i.e. $1/q' + 1/q = 1$.

Next we consider the case $k = n-1$: A basic result was obtained by E. M. Stein in the sixties. He discovered, for $n \geq 2$, that the Fourier

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transform has the following restriction property: For the unit sphere S^{n-1} in R^n and $d\sigma$ a rotationally invariant measure

$$\int_{S^{n-1}} |\widehat{f}(\xi)|^2 \, d\sigma(\xi) \leq C \, \|f\|^2_{L^{p'}(\mathbf{R}^n)}, \tag{2}$$

for some $p' > 1$. By localizing to a ball of radius λ in $\mathbf{R}^n$, the dual of this inequality states that the operator T_λ with phase function $\varphi(x,\xi) = x \cdot \psi(\xi)$, where $\psi : U \to S^{n-1}$ parameterizes a coordinate neighborhood of the unit sphere in $\mathbf{R}^n$, has norm of order $\lambda^{-n/p}$ as a map from $L^2(U)$ to L^p_{loc} for some $p < \infty$. Improvements on the range of exponents p were made by P. Tomas [26] and E.M. Stein. Moreover, it was shown by E.M. Stein (see [24]) that for nonlinear phase functions φ the norm of T_λ has order $\lambda^{-n/p}$ as an operator from L^2_{comp} to L^p_{loc} for $p \geq 2(n+1)/(n-1)$, provided φ satisfies the following curvature condition: for each $x \in \mathbf{R}^n$ the hypersurface parameterized by

(3) $\xi \mapsto \nabla_x \varphi(x,\xi)$ *has nonvanishing Gaussian curvature.*

This (L^2, L^p)-result is sharp in the sense that $p = 2(n+1)/(n-1)$ is critical. Moreover, due to an example of J. Bourgain [2, 4] under the conditions (3) and (1) Stein's result can not be improved in case n is odd if we require $q = \infty$ (see also [15]).

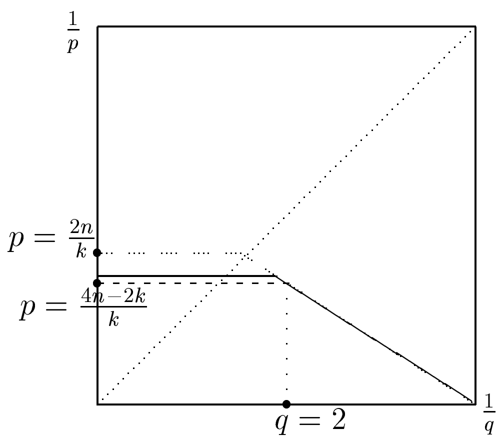

Figure: (p,q)-range for $k = n$ and $k = n-1$.

However under some further conditions, remarkable improvements have been made by J. Bourgain [1]. His method, which led to further improvements in [27, 5] and [25], showed in particular for the situation of the unit sphere described above, that for certain exponents p less then the critical L^2-exponent $2(n+1)/(n-1)$ that $\|T_\lambda\|_{L^\infty(U)\to L^p_{loc}} \leq C\,\lambda^{-\frac{n}{p}}$. It is expected that the (q,p)-range for which this inequality holds is determined by: $p > 2n/k$ and $p \geq (2n-k)/kq'$ (see Figure). For $n = 2$ the norm of T_λ is essentially well understood due to work of L. Carleson and P. Sjölin [6] provided the curvature condition (1) and (3) are satisfied. We note that for the expected bounds the crucial point is to understand for the operators T_λ the (L^∞, L^p)-bounds.

Our main concern here are the cases $k < n-1$. There have been some results in the past addressing the problem of (L^q_{comp}, L^p_{loc})-bounds for oscillatory integral operators in these cases. However, these results mainly discussed the cases $k = 1, n-2$ or $n/2$ for n even (see e.g. [7], [8], [12], [11], [19], [20]). For different k some results are obtained in [10] and [17]. A natural question which we ask here is the following: *Suppose* (1) *holds. Under which conditions on the phase function* φ *does* T_λ *map* $L^q_{comp}(\mathbf{R}^k)$ *to* $L^p_{loc}(\mathbf{R}^n)$ *with norm of order* $\lambda^{-\frac{n}{p}}$ *in the full range*

$$(4) \qquad p \geq \frac{2n-k}{k} q' \qquad \text{and} \qquad p > \frac{2n}{k} \qquad \text{for} \quad k < n?$$

One of our results will be that we can expect these optimal bounds only when $k \geq n/2$. We will also see that in some situations where the phase function is linear in the x-variables an analogue of the Stein-Tomas result holds, i.e. optimal (L^2, L^p)-bounds hold, but the (L^∞, L^p)-bounds fail to hold in the range given in (4). This apparently appears only if $k < n-2$.

We should mention that one of the main difficulties which distinguishes the case $k < n-1$ from $k = n-1$ lies in the fact that, although a stationary phase argument shows that for $\psi \in C^\infty(\mathbf{R}^k)$ and most $x \in \mathbf{R}^n$ the decay of $T_\lambda \psi(x)$ is of order $\lambda^{-k/2}$, in general isotropic bounds for $T_\lambda f(x)$ decay slower (see e.g. [9]).

2. A necessary curvature condition

Here we derive a necessary condition on the phase function φ such that T_λ is bounded in the full range described in the above figure (for $k \leq n$). First we observe that if T_λ has norm of order $\lambda^{-n/p}$, then for each x_0 the operator with phase function $\varphi_R(x,\xi) = R\,(\varphi(x_0 + x/R, \xi) - \varphi(x_0,\xi))$ satisfies the same bounds uniformly in $R > 0$. Hence, the operator with the linearized phase function $(x,\xi) \mapsto x \cdot \nabla_x \varphi(x_0,\xi)$ has the same bounds. By reparameterizing the k-dimensional submanifold $\xi \mapsto \nabla_x \varphi(x_0,\xi)$ over the tangent plane at a given point using translation invariance we may assume that $x \cdot \nabla_x \varphi(x_0,\xi)$ has the form $(x_1 \cdot \xi, x_2 \cdot \psi(\xi))$, with $\psi(0)$ and $d_\xi \psi(0)$ both vanishing. A further scaling argument –replacing $x_1 \to R x_1, x_2 \to R^2 x_2$ and $\xi \to \xi/R$ and letting $R \to \infty$– shows that the phase function $x_1 \cdot \xi + x_2 \cdot \tilde{\psi}(\xi)$, here $\tilde{\psi}$ is the second order part of the Taylor expansion of ψ, gives rise to an operator

$$\tilde{T} f(x) = \int_{\mathbf{R}^k} e^{i(x_1 \cdot \xi + x_2 \cdot \tilde{\psi}(\xi))}\, f(\xi)\, e^{-|\xi|^2/2}\, d\xi,$$

which is bounded from $L^q(\mathbf{R}^k)$ to $L^p(\mathbf{R}^n)$ for (q,p) on the line $p = (2n-k)/kq'$ provided that T_λ has norm of order $\lambda^{-n/p}$ on this line.

Write

$$x_2 \cdot \tilde{\psi}(\xi) = \frac{1}{2}\, \xi \cdot Q(x_2)\xi,$$

with $Q(x_2) = \sum_{j=1}^{n-k} x_{2,j}\, B_j$ and $B_j \in \mathrm{Sym}(k)$, where $\mathrm{Sym}(k)$ denotes the space of symmetric matrices on $\mathbf{R}^k$. To emphasize the dependence on Q the operator $\tilde{T}$ will be denoted by T_Q in the following and we refer to the submanifolds parameterized by

$$H : \mathbf{R}^k \ni \xi \mapsto (\xi, \xi \cdot B_1\xi, \ldots, \xi \cdot B_{n-k}\xi) \in \mathbf{R}^n$$

as the associated quadratic submanifold M_Q.

If T_Q maps $L^\infty(\mathbf{R}^k)$ to $L^p(\mathbf{R}^n)$, then in particular for the constant function 1 we have $G = \tilde{T}_Q 1 \in L^p$. A computation gives

$$\|G\|_p^p = C \int_{\mathbf{R}^{n-k}} |\det(E + iQ(x_2))|^{-p/2+1}\, dx_2, \tag{5}$$

here E denotes the unit matrix in $\mathrm{Sym}(k)$. To ensure that the above integral is finite for some $p < \infty$ we need that the symmetric matrices $B_j, 1 \leq j \leq n-k$, are linearly independent which requires that $n \leq k(k+3)/2$. To find a further restriction we show

Proposition 2.1. *If the function G above is in $L^p(\mathbf{R}^n)$, for all $p > 2n/k$, then*

$$\int_{S^{n-k-1}} |\det Q(x)|^{-\gamma}\, d\sigma(x) < \infty \quad \textit{for all } \gamma < \frac{n-k}{k}, \tag{6}$$

with σ the uniform measure on the unit sphere S^{n-k-1}.

Proof. To see this we use polar coordinates in (5) and write $x_2 = ry$, and $r = |x|$. Then

$$\begin{aligned} |\det(E + iQ(x))|^2 &= \det(E + Q(x)^2 \\ &= 1 + r^2c_1^2 + \ldots c_{k-1}^2 r^{2k-2} + \det Q(y)^2 r^{2k}. \end{aligned}$$

Suppose that $\sup_{j,y} |c_j(y)| \leq c$ and let $L(y) = \max\{1, c/|\det Q(y)|\}$. Then we get the following lower bound on $\|G\|_p^p$ for $p = 2n/k + 2\epsilon$, $\epsilon > 0$:

$$\int_{S^{n-k-1}} \int_{L(y)}^{\infty} \frac{r^{n-k-1}}{|r^k \det Q(y)|^{(n-k)/k+\epsilon}}\, dr d\sigma(y),$$

which evaluates to (6) by integrating the inner integral. □

As a consequence we show:

Corollary 2.2. *Suppose the function G defined above is in $L^p(\mathbf{R}^n)$ for all $p > 2n/k$. Then the following hold:*

- *If k is odd, then $k \geq \frac{n}{2}$.*

- *If k is even and the subspace $\{Q(x) | x \in \mathbf{R}^{n-k}\}$ intersects the cone of positive definite matrices in* $\mathrm{Sym}(k)$, *then* $k \geq n/2$.

Proof. The idea here is to find a hypersurface on the unit sphere in $\mathbf{R}^{n-k}$ where the function $\det Q$ vanishes at least of order 1. Assuming $k < n/2$, i.e. $(n-k)/k > 1$, Proposition (2.1) implies that the integral $\int_{S^{n-k-1}} |\det Q|^{-1} \, d\sigma$ is finite. Since the polynomial $\det Q(x) = \det(x_1 B_1 + \cdots + x_{n-k} B_{n-k})$, with $B_i \in \mathrm{Sym}(k)$, is homogenous of degree k for some power $\alpha > 0$ the function $|x|^{\alpha} |\det Q(x)|^{-1}$ must be integrable over the unit ball in $\mathbf{R}^{n-k}$. We can assume that $\det Q(x)$ does not vanish identically and that B_1 is a diagonal matrix with entries ± 1. If k is odd we can write locally $\det Q(x) = (x_1 - \varphi(x_2, \ldots, x_{n-k}))\psi(x)$ where φ, ψ are real continuous functions and $\varphi(0) = 0$. Hence , for all $\alpha > 0$, $|\det Q|^{-1}$ is not locally integrable on the unit ball in $\mathbf{R}^{n-k}$ and therefore $k \geq n/2$. To show the second part we may assume that $B_1 = E$. Then $x_1 \to Q(x_1, x_2, \ldots, x_{n-k})$ is the characteristic polynomial of the symmetric matrix $Q(0, x_2, \ldots, x_{n-k})$ and therefore has only real zeros. So again $\det Q(x) = (x_1 - \varphi(x_2, \ldots, x_{n-k}))\psi(x)$. As before we find that k has to be $\geq n/2$. □

The condition in the proposition above may be phrased in an invariant way. Consider the submanifold M parameterized by $\xi \mapsto \nabla_x \varphi(x_0, \xi)$ and fix a point $P = \nabla_x \varphi(x_0, \xi_0)$. We assume that M carries the induced Euclidean metric. Let $N_P(M)$ be the normal plane at $P \in M$, $T_P(M)$ be the tangent plane at P, $v \in N_P(M)$ and let $G_P(v)$ be the Gaussian curvature at P of the orthogonal projection of M (along v) into $\mathbf{R}v \oplus T_P(M)$. Then (6) states that

$$(7) \qquad \int_{S^{n-k-1} \subset N_p(M)} |G_P(v)|^{-\gamma} \, d\sigma(v) \; < \; \infty \quad \text{for all} \quad \gamma < \frac{n-k}{k},$$

where σ denotes a nontrivial rotationally invariant measure on the unit sphere in $N_P(M)$.

3. Restriction to quadratic submanifolds

In the following we show some positive results for the operators T_Q. We write $T_Q f = \widehat{f \, d\mu_Q}$, where $d\mu_Q$ is the measure on $\mathbf{R}^n$ with support on M_Q defined by

$$\mu_Q(f) = \int_{\mathbf{R}^k} f(\xi, H(\xi)) \, e^{-|\xi|^2/2} \, d\xi.$$

Theorem 3.3. *If* $\int_{S^{n-k-1}} |\det Q(x)|^{-\gamma} \, d\sigma(x) < \infty$ *for* $\gamma = \frac{n-k}{k}$. *Then* T_Q *is bounded from* $L^2(\mathbf{R}^k)$ *to* $L^p(\mathbf{R}^n)$ *for* $p \geq 2\frac{2n-k}{k}$.

Proof. It is enough to show (and in fact equivalent) that the composition $T_Q T_Q^* f = \widehat{d\mu} * f$ maps the dual space $L^{p'}(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$ for $p \geq 2\frac{2n-k}{k}$. Our strategy is now to define an analytic family T_z which evaluates at $z = 0$ to T_Q and is bounded from L^1 to L^∞ on the line $\Re z = 1/2$ and from L^2 to L^2 for $\Re z = -\frac{n-k}{k}$. A complex interpolation argument will then give the theorem. This is analogous to Stein's proof of the Tomas-Stein theorem. The main point here is to find a suitable analytic family. To define this analytic family we split variables and write as in the previous section $x = (x_1, x_2) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$. For $z \in \mathbf{C}$ we put

$$K_z(x) = \frac{(1 + |\det Q(x_2)|)^z}{,\,(n-k+kz)} \widehat{d\mu_Q}(x_1, x_2).$$

A computation shows that the latter expression is a constant multiple of

$$\frac{(1 + |\det Q(x_2)|)^z}{,\,(n-k+kz)} \det(E + iQ(x_2))^{-1/2}\ e^{-x_1\cdot(E+iQ(x_2))^{-1}x_1/2} \tag{8}$$

We define $T_z f = K_z * f$. Note that $T_0 f = c\,\widehat{d\mu} * f,\ c \neq 0$, and the family T_z is analytic in the whole complex plane. For (L^1, L^∞)-bounds for T_z we have to get uniform bounds for K_z on $\Re z = 1/2$. This follows easily from $(1 + |\det Q(x_2)|)^2 \leq \det(E + Q(x_2)^2)$. For the L^2-boundedness we have to bound the Fourier transform of K_z. To compute the Fourier transform of K_z we first evaluate the Fourier transform with respect to the x_1-variable. This gives

$$\widehat{K_z}(\xi_1, \xi_2) = C \int_{\mathbf{R}^{n-k}} e^{-ix_2\cdot\xi_2}\ \frac{(1 + |\det Q(x_2)|)^z}{,\,(n-k+kz)}\ e^{-\xi_1\cdot(E+iQ(x_2))\xi_1/2}\ dx_2$$

Hence, to bound $\widehat{K_z}$ it is enough to get bounds on the Fourier transform of $(1 + |\det Q|)^z$. Now, for $\Re z = -\frac{n-k}{k} + \varepsilon, \varepsilon > 0$, we find, using polar coordinates $x_2 = ry, r = |x_2|$ the following bound for $\|\widehat{K_z}\|_\infty$:

$$C \sup_{\eta \in \mathbf{R}^{n-k}} \left| \int_{S^{n-k-1}} \int_0^\infty \frac{(1 + r^k |\det Q(y)|)^z}{,\,(n-k+kz)}\ e^{iry\cdot\eta}\ r^{n-k-1}\ dr\ dy \right|$$

Since we are assuming $\int_{S^{n-k-1}} |\det Q(y)|^{-\frac{n-k}{k}}\ d\sigma(y) < \infty$, we see that the above integral is bounded by a constant times

$$\sup_{x \in \mathbf{R}} \left|\frac{1}{,\,(n-k+kz)}\ \int_0^\infty r^{n-k-1}(1 + r^k)^z\ e^{ixr}\ dr\ \right|.$$

On the line $\Re z = -\frac{n-k}{k}$, the function $r^{n-k-1}(1 + r^k)^z$ is essentially $r^{-1+kz+n-k}$, i.e., homogeneous of degree $-1 + is$. Its Fourier transform is homogeneous of degree $-is$ and produces a pole at $z = -\frac{n-k}{k}$ which cancels with the Gamma function in front of the last integral. Hence $|\widehat{K_z}|$ is bounded. □

We note that if we would have been working with the analytic family

$$\tilde{K}_z(x_1,x_2) = \frac{1}{,\,(n-k+kz)}\ \det(E+Q(x_2)^2)^{z/2}\ \widehat{d\mu}(x_1,x_2)$$

then the above method gives the following

Corollary 3.4. *If $\int_{S^{n-k-1}} |\det Q(x)|^{-\gamma}\ d\sigma(x) < \infty$ for all $\gamma < \frac{n-k}{k}$. Then T_Q is bounded from $L^2(\mathbf{R}^k)$ to $L^p(\mathbf{R}^n)$ for $p > 2\frac{2n-k}{k}$.* □

Arguing similarly one can show that if for a suitable polynomial $p(z)$ the ζ-distributions

$$\zeta_z(f) = p(z)\int_{\mathbf{R}^{n-k}} |\det Q(y)|^z\ f(y)\ dy,$$

has a bounded Fourier transform on the line $\Re(z) = -\frac{n-k}{k}$ then T_Q is bounded from $L^2(\mathbf{R}^k)$ to $L^p(\mathbf{R}^n)$ (p has only to annihilate finitely many poles of $\widehat{\zeta_z}$). Using this observation one can show that in certain cases one has optimal (L^2, L^p)-bounds for T_Q, although the (L^∞, L^p)-bounds fail to hold for some $p > 2n/k$. We provide a few examples in the following.

First we define for $(x,X) \in \mathbf{R}^k \times \mathrm{Sym}(k) \cong \mathbf{R}^n$ with $n = \frac{k(k+3)}{2}$

$$Tf(x,X) = \int_{\mathbf{R}^k} e^{i(x\cdot\xi+\xi\cdot X\xi)}\ f(\xi)\ e^{-|\xi|^2/2}\ d\xi. \tag{9}$$

Then we have the following theorem, whose first part was independently shown in [10] and for the special case $k=2$ in [7].

Theorem 3.5. *The operator T has the following properties:*

1. *T is bounded from $L^2(\mathbf{R}^k)$ to $L^p(\mathbf{R}^n)$ iff $p \geq 2\frac{2n-k}{k}$.*
2. *T is unbounded as an operator from $L^\infty(\mathbf{R}^k)$ to $L^p(\mathbf{R}^n)$ for $p \leq 2(k+1)$ $(= \frac{2n}{k} + k - 1)$.*

For the proof we will need the Fourier transform of $|\det X|^z, X \in \mathrm{Sym}(k), z \in \mathbf{C}$. This has been computed first by T. Shitani and more recently by Faraut and Satake [13] using the theory of Jordan algebras. To state the result, we note first that $\mathrm{Sym}(k) \cap GL(k,\mathbf{R})$ decomposes under the operation $(g,X) \to gXg^t$ into $k+1$ $GL(k,\mathbf{R})$-orbits, $\Omega_j, j = 0,\ldots,k$, where Ω_j is the cone of symmetric matrices of signature $(k-j,j)$. Let Ω_0 be the orbit of the unit matrix $E \in \mathrm{Sym}(k)$. Associated to Ω_0 is the Gamma function

$$\Gamma_{\Omega_0}(s) = \int_{\Omega_0} e^{-trace(X)} \, (\det X)^{s-\frac{k+1}{2}} \, dX$$
$$= (2\pi)^{\frac{k(k-1)}{4}} \prod_{0 \le j \le k} \Gamma\,(s - \frac{j-1}{2}).$$

For $0 \le i \le k$ we define Zeta distributions

$$\zeta_i(f, s) = \int_{\Omega_i} f(X) \, |\det X|^s \, dX.$$

The poles here lie on the arithmetic progression $\frac{1}{2}\mathbf{Z} \cap (-\infty, -1]$. We have

$$\zeta_i(\widehat{f}, s - \frac{k+1}{2}) = (2\pi)^{-k(k+1)/2} \, e^{i\pi k s/2} \, \Gamma_{\Omega_0}(s) \sum_{0 \le i \le k} u_{i,j}(s) \, \zeta_j(f, -s),$$

where $u_{i,j}$ is a polynomial of degree k in $e^{-i\pi s}$. Putting $s = \frac{k+1}{2}(1-z)$, then it easily follows that the Fourier transform of

$$\frac{1}{\Gamma_{\Omega_0}(\frac{k+1}{2}(1-z))} \quad \frac{1}{|\det X|^{z\frac{k+1}{2}}}$$

is a bounded function in $X \in \mathrm{Sym}(k)$ on the imaginary line $\Re z = 1$ with bounds growing at most exponentially along this line. Hence part (i) follows. For the second part we will show that $\|T1\|_p < \infty$ if and only if $p > 2(k+1)$. In fact, since $\|T1\|_p^p = C \int_{\mathrm{Sym}(k)} |\det(E+iX)|^{-p/2+1} \, dX$, we find using generalized polar coordinates

$$\int_{\mathbf{R}^k} \prod_{1 \le j \le k} |1 + i\lambda_j|^{-p/2+1} \prod_{1 \le i < j \le k} |\lambda_i - \lambda_j| \, d\lambda_1 \dots d\lambda_k < \infty$$

Now, the worst decay of the integrant is along the coordinate axes. Checking exponents it follows that the last integral is finite if and only if $p > 2(k+1)$. □

We remark that one can show that the operator (9) is a bounded operator from $L^\infty(\mathbf{R}^k)$ to $L^{2k+2}(B_R)$ with norm of order $(\log R)^{\frac{1}{2k+2}}$, where B_R is a ball of radius R in $\mathbf{R}^n$ (note that $2k+2$ is an even integer).

As a second example we consider for $m > 1$ the set $M(m, \mathbf{C})$ of complex $m \times m$-matrices which we might consider as a real subspace of $\mathrm{Sym}(4m)$ via the following real linear map

$$Q : M(m, \mathbf{C}) \ni Z = X + iY \to \begin{pmatrix} 0 & 0 & X & Y \\ 0 & 0 & -Y & X \\ {}^tX & -{}^tY & 0 & 0 \\ {}^tY & {}^tX & 0 & 0 \end{pmatrix} \in \mathrm{Sym}(4m),$$

where X, Y denote the real resp. imaginary part of Z. Note that for $\lambda \in \mathbf{C}$ we have $\det(\lambda E + iQ(Z)) = \det(\lambda^2 E + Z^* Z)^2$. It has been shown by E.M. Stein [23] that the Fourier transform of the Zeta distribution

$$\zeta(f, s) = \int_{M(m,\mathbf{C})} |\det Z|^s \; f(Z) dZ$$

is given by the $\frac{1}{\gamma_*(s)}\ |\det Z|^{-s-2m}$, where $\gamma_*(s) = \gamma(s)\gamma(s-2)\ldots\gamma(s-2m+2)$, $\gamma(s) = \, ,(-\frac{s}{2})/,(\frac{2+s}{2})$. Let $k = 4m$ and define for $(x, Z) \in \mathbf{R}^k \times M(m, \mathbf{C}) \cong \mathbf{R}^n$, with $n = 2m(m+2)$, the oscillatory integral operator

$$Tf(x, Z) = \int_{\mathbf{R}^k} e^{i(x\cdot\xi + \xi\cdot Q(Z)\xi))} \; f(\xi) \; e^{-|\xi|^2/2} \; d\xi.$$

Then we have

Theorem 3.6. *The operator T has the following properties:*

1. *T is bounded from $L^2(\mathbf{R}^k)$ to $L^p(\mathbf{R}^n)$ iff $p \geq 2\frac{2n-k}{k}$.*
2. *T is unbounded as an operator from $L^\infty(\mathbf{R}^k)$ to $L^p(\mathbf{R}^n)$ for $p \leq 2m+1$ $(= \frac{2n}{k} + \frac{k}{4} - 1)$.*

Using polar coordinates associated to the Cartan decomposition corresponding to the symmetric space $SU(n,n)/S(U(n) \times U(n))$ it is not hard to check that we have $T1 \in L^p$ iff

$$\int_{\mathbf{R}^m} \frac{h_1 \ldots h_m \prod_{1\leq i<j\leq m}(h_i^2 - h_j^2)^2}{\prod_{1\leq i\leq m}(1+h_i^2)^{p-2}} dh_1 \ldots dh_m$$

is finite, i.e. $p > 2m+1$. This confirms the second part of the theorem.

These examples suggest that sharp L^2-restriction estimates should hold for *most* quadratic submanifolds. It would be interesting to find out for which sets inside $\mathrm{Sym}(k)^{n-k}$ for which sharp L^2-restriction fails (so far we have only some insight in the case $n-k = 2, 3$).

In the above examples k was always $< n/2$. However, there even in case $k = n-3$ examples for which we have optimal (L^2, L^p)-bounds but the (L^∞, L^p)-bounds fail for some $p > 2n/k$: An example is provided by

$$\begin{aligned}\xi \cdot Q(x)\xi = (x_1 + x_3)\xi_{n-3}^2 + x_1(\xi_1^2 + \xi_3^2 + \cdots + \xi_{n-5}^2)x_3(\xi_2^2 + \xi_4^2 + \ldots \xi_{n-4}^2) \\ + 2x_2(\xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{n-5}\xi_{n-4})\end{aligned}$$

It can be shown that the Fourier transform of the corresponding ζ-distribution is essentially a cone multiplier of order $3/(2n-6)$ hence bounded function. But we do not have (L^∞, L^p)-boundedness for all $p > 2n/(n-3)$. To see this one has to check when $|\det(E + iQ(x))|^{-p/2+1}$ is integrable, where Q is the Hessian of the quadratic form $\xi \to q(x,\xi)$. Now, $E + iQ(x_1 - x_3, x_2, x_1 + x_3)$ has eigenvalues $1+2ix_1, 1+i(x_1 \pm \sqrt{x_2^2 + x_3^2})$ and we find using polar coordinates in the x_2, x_3-variables that the L^1-norm of $|\det(E+iQ(x))|^{-p/2+1}$ is bounded from below by a multiple of

$$\int_{x_1>0} \frac{1}{(1+|x_1|)^{\frac{p}{2}-1}} \int_{|x_1-r|\leq 1} \frac{r\ dr}{\Big((1+|x_1+r|)^{\frac{n-4}{2}}(1+|x_1-r|)^{\frac{n-4}{2}}\Big)^{\frac{p}{2}-1}}\, dx_1.$$

The last integral is finite iff $\frac{p}{2} - 1 + \frac{n-4}{2}(\frac{p}{2} - 1) > 2$, i.e. $p > 2\frac{n+2}{n-2}$. For more details and a description of how this examples arises in the context of nonregular orbits under certain Lie group actions we refer to [16]. Finally we mention the following theorem for the case $k = n-2$ (see [16] and [7]).

Theorem 3.7. *If $B_1, B_2 \in \mathrm{Sym}(n-2)$ are linear independent then for the operator T_Q corresponding to $Q(x_1,x_2) = x_1B_1 + x_2B_2$ the following statements imply each other*

1. *T_Q is bounded from $L^2(\mathbf{R}^k)$ to $L^p(\mathbf{R}^n)$ for $p \geq 2\frac{n+2}{n-2}$.*
2. *$T_Q 1 \in L^p(\mathbf{R}^n)$ for $p > \frac{2n}{n-2}$.*
3. *$\int_{S^1} |\det Q(x)|^{-\gamma} < \infty$ for $\gamma < \frac{2}{n-2}$.*

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