STRICHARTZ ESTIMATES AND LOCAL WELLPOSEDNESS FOR THE SCHRÖDINGER EQUATION WITH THE TWISTED SUB-LAPLACIAN

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Abstract. We obtain Strichartz estimates for the linear Schrödinger equation associated with the twisted sub-Laplacian on \( \mathbb{C}^n \). As a consequence, we prove the local wellposedness for semilinear Schrödinger equation with polynomial nonlinearity in certain magnetic field.

1. Introduction and main results

As is well-known, the Strichartz estimates play an important role in the study of wellposedness theory for nonlinear dispersive equations [9, 11]. In this paper we are concerned with proving the Strichartz estimates for the twisted Laplacian on \( \mathbb{C}^n \) and finding applications to the associated semilinear NLS.

The twisted Laplacian \( L \) on \( \mathbb{C}^n \) is given by

\[
L = -\frac{1}{2} \sum_{i=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j),
\]

where \( Z_j = (\frac{\partial}{\partial z_j} + \frac{1}{2} \bar{z}_j), \bar{Z}_j = (\frac{\partial}{\partial \bar{z}_j} - \frac{1}{2} z_j), \) \( j = 1, \ldots, n \), are 2n vector fields on \( \mathbb{C}^n \). For \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), writing \( z_j = x_j + iy_j \) and its conjugate \( \bar{z}_j = x_j - iy_j \).

Then we can also write \( L \) on \( \mathbb{R}^n \times \mathbb{R}^n \) as

\[
L = -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2 + |y|^2) - i \sum_{j=1}^{n} (x_j \partial_{y_j} - y_j \partial_{x_j})
\]

\[
= -\sum_{j=1}^{n}(\partial_{x_j} - \frac{1}{2}iy_j)^2 + (\partial_{y_j} + \frac{1}{2}ix_j)^2,
\]

where \( x, y \in \mathbb{R}^n \). Thus it is a Schrödinger operator with constant magnetic potential [17], which can be viewed as a quantization of the motion of a charged particle (without spin) in a constant magnetic field, cf. Avron, Herbst, Simon et al [1] for physical background. The spectral theory of twisted Laplacian is well-known and intimately related to that of the sub-Laplacian on Heisenberg groups [25].

Let \( \hat{X}_j = \partial_{x_j} - \frac{1}{2}iy_j, \hat{Y}_j = \partial_{y_j} + \frac{1}{2}ix_j \). Then \( [\hat{X}_j, \hat{Y}_k] = i\delta_{jk} \). Using the Weyl representation \( (\mathbb{R}^{2n}, \pi) \)

\[
d\pi(\hat{X}_j) = -i\xi_j, \ d\pi(\hat{Y}_j) = \partial_{\xi_j},
\]

we have \( d\pi(L_a) = -\Delta_{R^n} + |\xi|^2 \), thus the spectrum of \( L \) is the set \( \sigma(L) = \{n+2k, \ k \in \mathbb{N} \} \) and each eigenspace \( E_k \) has infinite dimensions.
Consider the Schrödinger equation associated with $L$

$$i\partial_t u(t, z) - Lu(t, z) = F(t, z)$$

$$u(0, z) = f(z).$$

Motivated by the treatment in the Euclidean setting [9, 11], we will derive the Strichartz estimates from the dispersive estimates and energy conservation. Similar considerations have been given in [2, 8, 16, 10] for variants of the sub-Laplacian on Heisenberg groups. Nandakumaran and Ratnakumar [16] obtained Strichartz estimates for the Hermite operator. Later Ratnakumar extended the result to the case of the special Hermite operator [19].

In $\mathbb{R}^n$, the Strichartz for the Cauchy problem (4) (i.e., $L = -\Delta$) reads

$$\left( \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}^n} |u(t, x)|^{\frac{2(n+2)}{n}} \ dx \right)^{\frac{n}{n+2}} \right)^{\frac{n+2}{n}} \leq C(\|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\frac{2(n+2)}{n+4}}(\mathbb{R}^{1+n})}).$$

This was generalized by Ginibre and Velo [9] for $L_1^q L_2^p$ norm for $(q, p)$ being an admissible pair when $q > 2$, and by Keel and Tao [11] when $q = 2$.

We say $(q, p)$ is an admissible pair on $\mathbb{C}^n$ if $\frac{2}{q} + \frac{2n}{p} = n$. Our first result is the following theorem.

**Theorem 1.1.** Let $(q, p)$ and $(\tilde{q}, \tilde{p})$ be admissible pair and $2 < q, \tilde{q} \leq \infty$, $2 \leq p, \tilde{p} < \frac{2q}{n-1}$. Let $T > 0$, $f \in L_1^q(\mathbb{C}^n)$ and $F(t, z) \in L_1^q([-T, T], L_1^p(\mathbb{C}^n))$. Then the solution $u(t, z)$ of (4) satisfies

$$\|u\|_{L_t^q([-T, T], L^p)} \leq C_{q,T}(\|f\|_{L^q} + \|F\|_{L_t^\tilde{q}([-T, T], L_p)}).$$

As in the classical cases [7, 5], the Strichartz inequality can be applied to show the local wellposedness for initial data with low regularity. In Section 4 we consider the Cauchy problem

$$i\partial_t u - Lu = F(u)$$

$$u(0, z) = f(z) \in W^{s, 2}_L,$$

where $F$ is a polynomial of order $m$, $F(0) = 0$, $W^{s, p}_L = L^{-s}(L^p(\mathbb{C}^n))$, the so-called twisted Sobolev spaces. We obtain

**Theorem 1.2 (LWP).** Let $s > \frac{n}{2} - \frac{1}{\max(m-1, 2)}$. For every bounded subset $\mathcal{B}$ of $W^{s, 2}_L$, there exists $T > 0$ such that for every initial data $f \in \mathcal{B}$ there exists a unique solution of (7)

$$u \in C([-T, T], W^{s, 2}_L) \cap L^q([-T, T], W^{s, p}_L),$$

where $(q, p)$ is an admissible pair with $q > \max(m-1, 2)$ and $p > n/s$. Moreover, the flow $f \mapsto u$ is Lipschitz from $\mathcal{B}$ to $C([-T, T], W^{s, 2}_L)$.

Magnetic NLS have been considered in Cazenave and Esteban [6], Yajima [26], Bouard [3], Nakamura [15], Michel [13] using Fourier integral operator methods. Also the Strichartz estimates were proved via PDE technique [12]. However, our method is based on special Hermite expansions and our result treats different non-linearity using modified Sobolev spaces.

The NLS generated by the twisted Laplacian may suggest the extension of our result to the NLS problem for the full sub-Laplacian on Heisenberg groups [2, 8], including the endpoint case [11, 23].

The remaining part of the paper is organized as follows. Section 2 is a brief summary of some basics regarding the special Hermite expansions. In Section 3 we prove the Strichartz estimates. Section 4 is devoted to the proof of the local wellposedness result.
2. Preliminary spectral theory for the twisted Laplacian

Let $H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})$, $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots \}$. The Hermite functions are given by $h_k(x) = (2^k k! \sqrt{\pi})^{-1/2} e^{-\frac{x^2}{2}} H_k$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_n^+$, define $\Phi(x) = \prod_{j=1}^{n} h_{\lambda_j}(x_j)$. Let $\alpha, \beta \in \mathbb{Z}_n^+$ and $z = x + iy \in \mathbb{C}^n$, we define the special Hermite functions on $\mathbb{C}^n$ as

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \Phi_{\alpha}(\xi + \frac{y}{2}) \Phi_{\beta}(\xi - \frac{y}{2}) d\xi. \quad (8)$$

It is easy to show that

$$\mathcal{L}(\Phi_{\alpha\beta}) = (2|\beta| + n) \Phi_{\alpha\beta},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then $\{ \Phi_{\alpha\beta} \}_{\alpha, \beta \in \mathbb{Z}_n^+}$ form a complete orthonormal system in $L^2(\mathbb{C}^n)$, see [25].

The special Hermite functions can be expressed in terms of Laguerre functions. Let $L_j^0(x), \ j \in \mathbb{Z}_+$ be the Laguerre polynomials of order $\alpha > -1$ defined using the generating function

$$\Sigma_{k=0}^{\infty} t^k L_j^0(x) = (1 - t)^{-\alpha - 1} \exp(\frac{xt}{1-t}). \quad (9)$$

Write $L_k(x) = L_j^0(x)$. According to the Mehler’s formula [25, Section 1.3, p.19], we have

$$\Phi_{\alpha\alpha}(z) = (2\pi)^{-\frac{n}{2}} \prod_{j=1}^{n} \L_{\alpha_j}(\frac{1}{2}|z|) \xi_j e^{-\frac{1}{4}|z|^2}. \quad (10)$$

The twisted convolution $f \times g$ on $\mathbb{C}^n$ is given by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z - \omega)g(\omega) e^{\frac{iz \cdot \omega}{4}} d\omega.$$ 

For $f \in L^2(\mathbb{C}^n)$ we can write the expansion in the following form

$$f(z) = (2\pi)^{-\frac{n}{2}} \Sigma_{\nu} f \times \Phi_{\nu\nu}(z) = (2\pi)^{-n} \Sigma_{k=0}^{\infty} f \times \varphi_k(z), \quad (11)$$

where $\varphi_k(z) = (2\pi)^{\frac{k}{2}} \Sigma_{|\nu| = k} \Phi_{\nu\nu}(z)$ coincide with the Laguerre functions $\varphi_k(z) = L_k^n(z) e^{-\frac{1}{4}|z|^2}$. Note that $(2\pi)^{-n} f \times \varphi_k$ is simply the projection of $f$ onto the eigenspace corresponding to the eigenvalue $2k + n$.

Indeed, from the relations [25, Proposition 1.3.2]

$$\Phi_{\mu\nu} \times \Phi_{\alpha\beta} = \begin{cases} (2\pi)^{\frac{1}{2}} \Phi_{\mu\beta} & \alpha = \nu \\ 0 & \alpha \neq \nu \end{cases}$$

we obtain

$$(2\pi)^{\frac{1}{2}} \Sigma_{\alpha} (f, \Phi_{\alpha\nu}) \Phi_{\alpha\nu} = f \times \Phi_{\nu\nu},$$

from which and $f(z) = \Sigma_{\alpha \beta} (f, \Phi_{\alpha\beta}) \Phi_{\alpha\beta}(z), (11)$ follows.

3. Linear estimates for Schrödinger equation

Consider the IVP (4) with $F = 0$:

$$i\partial_t u(t, z) - Lu(t, z) = 0, \quad u(0) = f \in L^2(\mathbb{C}^n). \quad (12)$$

The solution is given by

$$u(t, z) = e^{-itL} f(z) = (2\pi)^{-n} \Sigma_{k=0}^{\infty} e^{-it(2k+n)} f \times \varphi_k(z). \quad (13)$$

In fact, for each $t \in \mathbb{R}$,

$$\|e^{-itL} f(z)\|^2_{L^2} = (2\pi)^{-2n} \Sigma_{k=0}^{\infty} \|f \times \varphi_k(z)\|^2_{L^2} = \|f\|^2_{L^2}. \quad (14)$$
Since $L\varphi_k = (2k + n)\varphi_k$, we have that $u(t, z)$ satisfies (12) in weak $L^2$. Moreover, since $|e^{-it(2k+n)} - 1| \leq 2$, we have
\[
\|u(t, z) - f(z)\|_{L^2} \to 0 \quad \text{as } t \to 0,
\]
by a dominated convergence argument.

Let $K_t(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-it(2k+n)} \varphi_k(z)$. Write the special Hermite expansions of $u(t, z)$ in the form
\[
u(t, z) = f \times K_t(z).
\]
Then $\{e^{-itL}, t \in \mathbb{R}\}$ satisfy the semigroup property on $L^2$. Moreover, since $u(t + 2\pi, z) = u(t, z)$, the solution $u(t, z)$ is 2$\pi$-periodic in $t$.

In order to give the estimates of the semigroup $\{e^{-itL}, t \in \mathbb{R}\}$, we replace the parameter $it$ with $\gamma = r + it$, $r > 0$. Then the kernel of the semigroup $e^{-\gamma L}$ is given by
\[
K_{\gamma}(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)\gamma} \varphi_k(z).
\]

Using formula (9) we find
\[
K_{\gamma}(z) = (4\pi)^{-n} (\sinh(r + it))^{-n} e^{-\frac{1}{2}(\coth(r + it))|z|^2}.
\]

By the discussion above we easily see that for $f \in L^2$, $u_t(t, z) := e^{-\gamma L} f(z) = f \times K_{\gamma}(z)$ is the solution of IVP (12) with $u(0) = e^{-rL} f$.

Now we give the $L_p' - L_p$ estimate for the semigroup $\{e^{-\gamma L}, \gamma \in \mathbb{C}\}$.

**Lemma 3.1.** Let $r \geq 0, t \neq 0, 2 \leq p \leq \infty$ and $p' = p/(p - 1)$. Then
\[
\|e^{-(r+it)L} f(z)\|_{L^p} \leq e^{-rn} \|2\pi \sin t\|_{2^n(p - \frac{1}{2})} \|f\|_{L^{p'}}.
\]

**Remark.** We can also use the fact that $e^{-itL}$ has kernel
\[
(4\pi)^{-n} (i \sin t)^{-n} e^{-\frac{1}{2}\cot(t)|z|^2}
\]
to show the $L^1 \to L^\infty$ dispersive estimate, then the Strichartz follows as a corollary of [11].

**Proof.** First we prove the case $r > 0$. Since $\{\Phi_{\mu, \nu}\}$ is a complete orthonormal system in $L^2$, for $\gamma = r + it$, $r > 0$,
\[
\|u_t(t, z)\|_{L^2} = \left\| \sum_{\mu, \nu \in \mathbb{Z}_+^n} e^{-\gamma |\mu| + n} \langle f, \Phi_{\mu, \nu}\rangle \Phi_{\mu, \nu}\right\|_{L^2}
\]
\[
\leq e^{-rn} \left( \sum_{\mu, \nu \in \mathbb{Z}_+^n} \|\langle f, \Phi_{\mu, \nu}\rangle\|^2 \right)^{1/2} = e^{-rn} \|f\|_{L^2}.
\]

Note that
\[
\Re \coth(r + it) = \frac{1 - e^{-4r}}{1 + e^{-4r} - 2e^{-2r} \cos(2t)} \geq \frac{1 - e^{-2r}}{1 + e^{-2r}} > 0
\]
and
\[
|\sinh(r + it)| = |\sinh r \cos t + i \cosh r \sin t| \geq |\cosh r \sin t| \geq \frac{1}{2} e^r |\sin t|.
\]

We obtain
\[
\|u_t(z, t)\|_{L^\infty} = \|(f \times K_0)(z)\|_{L^\infty}
\]
\[
\leq (2\pi e^r |\sin t|)^{-n} \|f\|_{L^1}.
\]

Interpolating two inequalities (16) and (17) gives
\[
\|u_t(t, z)\|_{L^p} \leq (e^{-rn})^{2/p} (2\pi e^r |\sin t|)^{-2n(p - \frac{1}{2})} \|f\|_{L^{p'}}.
\]

\[
\leq e^{-nr} \|2\pi \sin t\|_{2^n(p - \frac{1}{2})} \|f\|_{L^{p'}}.
\]
The case $r = 0$ is a consequence of (18) by applying Fatou’s lemma and a density argument.

Now we prove Strichartz estimates for $u(t, z) = e^{-it\Delta}f(z)$. Let $2 \leq p \leq \frac{2n}{n-1}$. Recall that $(q, p)$ is called admissible on $\mathbb{C}^n$ if $\frac{2}{q} + \frac{2n}{p} = n$.

**Lemma 3.2.** Let $2 < q \leq \infty$, $2 \leq p < \frac{2n}{n-1}$ and $\frac{2}{q} + \frac{2n}{p} = n$. Let $u(t, z)$ be the solution to (12). Then for each $T > 0$, there exists a constant $C_{q,T} \leq C_q \max(1, T)$ such that

(a) \[ \|e^{itL}f(z)\|_{L^q([0,T], L^p)} \leq C_{q,T} \|f\|_{L^2} \] \hfill (19)

(b) \[ \|T_{T}^{*}F(t, z)dt\|_{L^2} \leq C_{q,T} \|F\|_{L^{p'}([0,T], L^{q'})}. \] \hfill (20)

**Proof.** We only need to show that inequality (b) holds for all $F$ in $L^{q'}([0,T], L^{p'})$ since (a) will then follow by duality. We follow the standard line of proof, the $TT^*$ argument for $e^{it\Delta}$ as in [11], see also [16]. Consider the bilinear form

$$ T(F, G) = \int_{-T}^{T} \int_{-T}^{T} \int_{\mathbb{C}^n} e^{itL}F(t, z)e^{isL}G(s, z)dzdsdt. $$

It is sufficient to show that for all $F, G$ in $L^{q'}([0,T], L^{p'})$

$$ |T(F, G)| \leq C_{q,T} \|F\|_{L^{p'}([0,T], L^{q'})} \|G\|_{L^{q'}([0,T], L^{p'})}. \] \hfill (21)

For $0 < T < \pi$, applying Lemma 3.1 with $1 \leq p' \leq 2$, we obtain

\[ \int_{\mathbb{C}^n} e^{itL}F(t, z)e^{isL}G(s, z)dz = \int_{\mathbb{C}^n} e^{i(t-s)L}F(t, z)G(s, z)dz \]
\[ \leq \|F(t, \cdot)\|_{L^{p'}}\|G(s, \cdot)\|_{L^{q'}} |\sin(t-s)|^{-2n\left(\frac{1}{p'} - \frac{1}{2}\right)}. \]

Since $\frac{2}{q} + \frac{2n}{p} = n$, applying the generalized Young inequality [20] gives

\[ |T(F, G)| \leq C_{q,T} \|F\|_{L^{p'}([0,T], L^{q'})} \|G\|_{L^{q'}([0,T], L^{p'})} |\sin t|^{-2n\left(\frac{1}{p'} - \frac{1}{2}\right)} |\sin s|^{-2n\left(\frac{1}{q'} - \frac{1}{2}\right)} \]
\[ \leq C_{q,T} \|F\|_{L^{p'}([0,T], L^{q'})} \|G\|_{L^{q'}([0,T], L^{p'})}, \quad 0 < T < \pi, \]

where we observe that the Young inequality requires that $1 < q < \infty$.

\[ |\sin s|^{-2n\left(\frac{1}{q'} - \frac{1}{2}\right)} \in L^{r, \infty}_{\text{loc}}, \]

$1/r = 1 + 1/q - 1/q' = 2/q = n(1 - \frac{2}{p})$ and $q > 2$.

For $T \geq \pi$, the estimate $C_{q,T} \leq C_{q,T}$ is a simple consequence of the periodic property of $u(t, z)$. This completes the proof of Lemma 3.2. \hfill \Box

**Remark.** Alternatively we can also prove Lemma 3.1 for $e^{-(r-i)t)L}F(t, z)$ first, and then use Fatou lemma plus a density argument to prove Lemma 3.2, cf. [19]. However it is more straightforward to prove the result as we proceed here for both lemmas.

Let $u(t, z)$ solve Equation (4). By Duhamel principle, $u$ is represented by

$$ u(t, z) = e^{-itL}f(z) - i \int_{0}^{t} e^{-(t-s)\Delta}F(s, z)ds. \] \hfill (22) $\]

**Proof of Theorem 1.1** In view of (22) and Lemma 3.2 we only need to show

\[ \|\int_{0}^{t} e^{-(t-s)\Delta}F(s, z)ds\|_{L^{q'}([0,T], L^{p'})} \leq C_{q,T} \|F\|_{L^{q'}([0,T], L^{p'})}. \] \hfill (23)
Define
\[ T(F,G) = \int_{-T}^{T} \int_{0}^{1} e^{i\sigma L} F(s, z) \overline{G(t, z)} dz ds dt. \]

By duality it is sufficient to prove the following bilinear estimate: For any two admissible pairs \((q, p), (\tilde{q}, \tilde{p})\), \(q \neq 2, \tilde{q} \neq 2, \)
\[ |T(F, G)| \leq C \| F \|_{L^q'([-T, T], L^{\tilde{p}'})} \| G \|_{L^{\tilde{q}}'([-T, T], L^p')}, \tag{24} \]
where \(C = C_{q, T} \leq C_q T\) is the same constant as in Lemma 3.2; in what follows we are going to impose the same conditions as here on the pairs \((q, p), (\tilde{q}, \tilde{p})\).

Let \(\chi_{(0,t)}(s)\) denote the characteristic function of \((0, t)\). By Lemma 3.2 we have for \(q > 2,\)
\[ \| e^{-itL} \int_{-T}^{T} e^{i\sigma L} (\chi_{(0,t)}(s) F(s, z)) ds \|_{L^2} \leq C \| F \|_{L^{\tilde{q}}'([-T, T], L^p')} \cdot \]

Thus by Fubini Theorem and Hölder inequality, we have
\[ |T(F, G)| \leq \sup_{t \in [-T, T]} \| \int_{0}^{1} e^{i\sigma L} F(s, z) ds \|_{L^2} \| G \|_{L^1([-T, T], L^2)} \leq C \| F \|_{L^{\tilde{q}}'([-T, T], L^p')} \| G \|_{L^1([-T, T], L^2)}. \]

On the other hand, (21) suggests that
\[ |T(F, G)| \leq C \| F \|_{L^{\tilde{q}}'([-T, T], L^p')} \| G \|_{L^1([-T, T], L^2)}. \tag{25} \]

Applying bilinear Riesz-Thorin interpolation, we obtain (24) for \((\tilde{q}, \tilde{p})\) with \(1 \leq \tilde{q}' \leq q', 2 \geq \tilde{p}' \geq p'.\) By symmetry (noting the symmetric form of the bilinear form \(T(F, G))\), write
\[ T(F, G) = \int_{-T}^{T} \int_{0}^{1} \left( \int_{-T}^{T} \chi_{(0,t)}(s) e^{i\sigma L} \overline{G(t, z)} ds \right) F(s, z) dz ds. \]

Repeating the same proof above we obtain for \(q' \leq \tilde{q}', p' \geq \tilde{p}',\)
\[ |T(F, G)| \leq C \| G \|_{L^{\tilde{q}}'([-T, T], L^p')} \| F \|_{L^{\tilde{q}'}([-T, T], L^{p'})}. \]

Thus we have proved that (24) holds for any admissible pairs \((q, p), (\tilde{q}, \tilde{p}), q \neq 2, \tilde{q} \neq 2.\) This completes the proof.

From (22), (14) and Theorem 1.1 we also have

**Corollary 3.3.** Let \(T > 0.\) Then the solution \(u(t, z)\) of (4) satisfies
\[ \| u \|_{C([-T, T], L^2)} + \| u \|_{L^q([-T, T], L^p)} \leq C_q T (\| f \|_{L^2} + \| F \|_{L^{\tilde{q}'}([-T, T], L^{p'})}), \]
where \((q, p), (\tilde{q}, \tilde{p})\) are admissible pairs with \(2 < q, \tilde{q} \leq \infty, 2 \leq p, \tilde{p} < \frac{2n}{n-1}.\)

4. **Semilinear Schrödinger equation**

In this section we consider the local wellposedness for the following Cauchy problem
\[ iu_t - Lu = F(u), \quad u(0, z) = f(z) \in W^{s, 2}_L, \tag{26} \]
where \(F\) is a polynomial of order \(m, F(0) = 0, W^{s, p}_L = L^{s, p}(L^p(C^n)) = \{ f = L^{-s}g : g \in L^p(C^n) \},\) the analogue of the usual Sobolev space, with \(\| f \|_{W^{s, p}_L} = \| g \|_{L^p(C^n)}.\)
As in the classical case, we can solve (26) by using the priori Strichartz estimates coupled with the Sobolev embedding theorem (Proposition 4.1).

The twisted Sobolev spaces were introduced in [18] and later used in [24] in the study of the spherical means for special Hermite expansions.

**Proposition 4.1.** Let \( s > n/p \) and \( 1 < p < \infty \). Then \( \mathcal{W}_L^{s,p} \hookrightarrow L^\infty(\mathbb{C}^n) \).

**Proof.** We only need to show that for \( n > s > n/p \) it holds that
\[
\|L^{-s}f\|_{L^\infty(\mathbb{C}^n)} \leq C\|f\|_{L^p(\mathbb{C}^n)}
\]
for all \( f \in L^2 \cap \mathcal{W}_L^{s,p} \). Let \( e^{-tL} \) be the heat kernel of \( L \), then for \( s > 0 \)
\[
L^{-s}f(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}e^{-tL}dt f(z).
\]
Since
\[
e^{-tL}f(z) = (2\pi)^{-n} \int_0^\infty e^{-t(2k+n)}f \times \varphi_k(z) = f \times p_t(z),
\]
where
\[
p_t(z) = (2\pi)^{-n} \int_0^\infty e^{-t(2k+n)}\varphi_k(z) = (4\pi t)^{-n}e^{-\frac{1}{2}(\coth t)|z|^2},
\]
it follows that the twisted convolution kernel of \( L^{-s} \) has the expression
\[
K^{-s}(z) = c_{s,n} \int_0^\infty t^{s-1}(\sinh t)^{-n}e^{-\frac{1}{2}(\coth t)|z|^2} dt.
\]
Note that if \( 0 < t \leq 1, \sinh t = O(t), \cosh t = O(1) \). Then it is easy to see that for \( 0 < s < n \),
\[
|K^{-s}(z)| \leq c \begin{cases} |z|^{2s-2n} & \text{if } |z| \leq 1, \\ e^{-c|z|^2} & \text{if } |z| > 1. 
\end{cases}
\]
We have for each \( q > 1 \)
\[
\int |K^{-s}(z)|^q dz \leq c \left( \int_{|z| \leq 1} |z|^{q(2s-2n)} dz + \int_{|z| > 1} e^{-cq|z|^2} dz \right) < \infty
\]
provided \( s > n - n/q \). Hence if \( n > s > n/p \), we obtain for all \( z \in \mathbb{C}^n \) and \( f \in L^2 \cap \mathcal{W}_L^{s,p} \),
\[
\|L^{-s}f(z)\| \leq \|K^{-s}\|_{L^q} \|f\|_{L^p},
\]
where \( 1/p + 1/q = 1 \). This proves the proposition. \( \square \)

**Remark.** The result agrees with the classical result since \( L \) is second order and \( \mathbb{C}^n \) has real dimension \( 2n \).

To show the LWP for (26) we will also need a “product rule” for fractional derivatives, namely, Proposition 4.7, whose proof depends on a few lemmas as we will see below.

Let us first establish the Littlewood-Paley inequality for \( L^p \). Fix \( \psi_0 \) and \( \psi \in C_0^\infty \) such that \( \psi_0, \psi \geq 0, \text{supp } \psi_0 \subset [0,1], \text{supp } \psi \subset [1/4,1] \) and \( \sum_{j=0}^\infty \psi_j^2(x) = 1 \) for all \( x \geq 0 \), where \( \psi_j(x) = \psi(2^{-j}x), j \geq 1 \).

**Lemma 4.2.** Let \( 1 < p < \infty \). Then there exists a positive constant \( C_p \) such that for all \( f \in L^p(\mathbb{C}^n) \),
\[
C_p^{-1}\|f\|_{L^p} \leq \|\left( \sum_{j=0}^\infty |\psi_j(L)f|^2 \right)^{1/2}\|_{L^p} \leq C_p\|f\|_{L^p}.
\]
(27)
The proof of Lemma 4.2 follows from the classical argument. Using multiplier theorem and Littlewood-Paley square function we know that the random function $m(ξ) := ±ψ(2^{-j}ξ)$, where $±$ are i.i.d. symmetric Bernoulli; are Mikhlin type multipliers uniformly in the choice of the signs $±$. Then (27) follows via Theorem 4.3 by applying Lemma 4.5, cf. [21, Chapter IV].

Consider the multiplier transform of the form

$$T_m f(z) = (2π)^{-n/2} \sum_{\nu \in \mathbb{Z}_+^n} m(\nu) f \times \Phi_{\nu \nu}(z).$$

For $k = 1, \ldots, n$, define $\Delta_k m(\nu) = m(\nu + e_k) - m(\nu)$, where $e_k = (0, \ldots, 1, \ldots, 0)$ with 1 in the $k$-th coordinate and 0’s elsewhere. If $β = (β_1, \ldots, β_n) \in \mathbb{Z}_+^n$, we define

$$\Delta^β m(\nu) = \Delta_{β_1} \cdots \Delta_{β_n} m(\nu).$$

We have the following multiplier theorem [25, 27].

**Theorem 4.3.** Let $m$ be a function defined on $\mathbb{Z}_+^n$ which satisfies

$$|\Delta^β m(\nu)| \leq C_n (1 + |\nu|)^{-|β|} \quad (28)$$

for all $β$ with $|β| \leq n + 1$. Then $T_m$ is bounded on $L^p(\mathbb{C}^n)$ for $1 < p < \infty$.

Let $χ_j(x) = χ(2^{-j}x)$, where $χ$ is a smooth cut-off function in $C_0^\infty$ with support in $[1/2, 2]$. Denote by $M_j$ the twisted convolution kernel of $T_{χ_j}$. The following weighted estimate holds according to [27, Lemma 2.1].

**Lemma 4.4.** There exists a constant $C_n$ such that for all $j \geq 0$,

$$\int_{\mathbb{C}^n} (1 + 2^j |z|^2)^{n+1} |M_j(z)|^2 dz \leq C_n 2^{nj}.$$

A simple consequence of Lemma 4.4 is that for all $j$ and all $f \in L^p \cap L^2$, $1 \leq p \leq \infty$ it holds that

$$\|χ_j(L)f\|_{L^p} \leq C\|f\|_{L^p}. \quad (29)$$

Recall the Rademacher functions from [21]. Let $r_m(t) = r_0(2^mt)$, where $r_0(t) = 1$, if $t \in [0, 1/2]$; –1 if $t \in (1/2, 1]$. The sequence of Rademacher functions are orthonormal (and mutually independent) over $[0, 1]$.

**Lemma 4.5.** Let $F(t) = \sum_{m=0}^{\infty} a_m r_m(t)$ and $\sum |a_m|^2 < \infty$. Then $F(t) \in L^p([0, 1])$ for each $p < \infty$. Moreover, there exist positive $c_p$ and $C_p$ such that

$$c_p\|F\|_p \leq \|F\|_2 = (\sum |a_m|^2)^{1/2} \leq C_p\|F\|_p.$$

The lemma above is contained in [21, Chapter IV, §5.2]. There are also included evident extensions to multi-dimensions.

**Proof of Lemma 4.2.** For $p = 2$, using $\sum_j \psi_j^2(x) = 1$ we have

$$\|\sum_{j=0}^{\infty} (\psi_j(L)f(z))^2\|_{L^2}^{1/2} = \sum_{j=0}^{\infty} (\psi_j(L)f, \psi_j(L)f) = \sum_{j=0}^{\infty} \sum_{\mu, \nu \in \mathbb{Z}_+^n} \psi_j^2(2|\nu| + n)(f, Ψ_{\mu \nu})^2 = \|f\|_{L^2}^2.$$

So by a standard duality argument, it suffices to prove the second inequality of (27). Let $m_t(x) = \sum_{j=0}^{\infty} r_j(t)ψ_j(x)$. We write

$$T_t f(z) = m_t(L)f(z) = (2π)^{-n} \sum_{k=0}^{\infty} m_t(2k + n)(f × φ_k)(z).$$
By the second inequality in Lemma 4.5, we have
\[
\left( \sum_{j=0}^{\infty} |\psi_j(L)f(z)|^2 \right)^{p/2} \leq C_p^p \int_0^1 |\sum_{j} \psi_j(L)f(z)\nu_j(t)|^p dt
\]
\[= C_p^p \int_0^1 |T_t f(z)|^p dt.
\]
Therefore, since \( m_t(\nu) := m_t(2|\nu| + n) \) satisfies (28), we obtain the desired estimate for \( 1 < p < \infty \)
\[
\int_{\mathbb{C}^n} \left( \sum_{j=0}^{\infty} |\psi_j(L)f(z)|^2 \right)^{p/2} dz \leq C_p^p \int_{\mathbb{C}^n} |f(z)|^p dz.
\]

\[\square\]

**Remarks.** From the proof one can easily see that the result remain valid if we only require \( \sum_j \psi_j^2(x) \approx 1 \).

An alternative proof of Lemma 4.2 would be to show the estimates \( L^1 \rightarrow \text{weak-} L^1(\ell^p) \) and \( L^1(\ell^p) \rightarrow \text{weak-} L^1 \), similar to the proof of vector-valued spectral multiplier theorem [17].

As a corollary to Lemma 4.2, the following norm characterization of \( W_{L}^{s,p} \) holds.

**Corollary 4.6.** Let \( 1 < p < \infty \) and \( s \geq 0 \). Then for all \( f \in L^p(\mathbb{C}^n) \), there exists a constant \( C_p \) such that
\[
C_p^{-1} \|f\|_{W_{L}^{s,p}} \leq \left\| \left( \sum_{j=0}^{\infty} 2^{2js} |\psi_j(L)f|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{W_{L}^{s,p}}.
\]

Let \( \Phi_j(x) = \sum_{j=0}^{\infty} \chi_{\nu}(x) \), \( j \geq 1 \). Using the decomposition
\[
f g = \sum_{ij} (\chi_i(L)f)(\chi_j(L)g)
\]
\[= \sum_i \Phi_i(L)g(\chi_i(L)f) + \sum_j (\chi_j(L)g)\Phi_j(L)f;
\]
and applying Corollary 4.6 and (29) we thus obtain the “product rule for fractional derivatives”.

**Proposition 4.7.** Let \( 1 < p < \infty \) and \( s \geq 0 \). Then for all \( f, g \in L^\infty \cap W_{L}^{s,p} \),
\[
\|fg\|_{W_{L}^{s,p}} \leq C(\|f\|_{L^\infty} \|g\|_{W_{L}^{s,p}} + \|f\|_{W_{L}^{s,p}} \|g\|_{L^\infty}).
\]

We are now ready to prove the local existence and uniqueness of (26).

**Proof of Theorem 1.2.** By Duhamel principle we consider the mapping
\[
\Phi(u)(t) = e^{itL}f - i \int_0^t e^{i(t-\tau)L}F(u(\tau))d\tau
\]
on the space \( X_T = C([-T,T], W_{L}^{s,2} \cap L^q([-T,T], W_{L}^{s,p}) \), which is endowed with the norm
\[
\|u\|_{X_T} = \max_{|\tau| \leq T} \|u(\tau)\|_{W_{L}^{s,2}} + \|u\|_{L^q([-T,T], W_{L}^{s,p})}.
\]
Let \( \mathcal{B} = \{ u \in X_T : \|u\|_{X_T} \leq \gamma \} \), where \( \gamma \) is a constant to be chosen later. Define the metric \( \rho(u, v) := \|u - v\|_{X_T} \). Then \( (\mathcal{B}, \rho) \) is a (convex) close set. We will show that \( \Phi \) is a contraction mapping in \( (\mathcal{B}, \rho) \). According to Lemma 3.2 and Proposition
4.7, we have
\[ \| \Phi(u) \|_{X_T} \leq C(\| f \|_{W^{s,2}} + \int_{-T}^{T} |F(u(\tau))|_{W^{s,2}} d\tau) \]
\[ \leq C(\| f \|_{W^{s,2}} + \int_{-T}^{T} (1 + \| u(\tau) \|_{L^{m-1}}^{m-1}) \| u(\tau) \|_{W^{s,2}} d\tau), \]
where in the first step we have used the property that \( L^s \) and \( e^{itL} \) commute. Now we can take \( q > \max(m-1,2) \) and take \( p \) to be the corresponding Strichartz index satisfying \( 1/p = 1/2 - 1/(mq) \). These are the numbers chosen in the definition of the space \( X_T \). Finally, we conclude the argument as follows: Proposition 4.1 tells that
\[ \| u(\tau) \|_{L^\infty} \leq C \| u(\tau) \|_{W^{s,p}}, \]
where \( s > n/p = n/2 - 1/q > n/2 - 1/\max(m-1,2) \). Let \( r = 1 - \frac{m-1}{q} \). Applying Hölder inequality in \( \tau \) we obtain
\[ \| \Phi(u) \|_{X_T} \leq C(\| f \|_{W^{s,2}} + C(T)\| u \|_{X_T} + T^r \| u \|_{X_T^r}). \]
Similarly we have
\[ \| \Phi(u) - \Phi(v) \|_{X_T} \leq CT^r(1 + \| u \|_{X_T} + \| v \|_{X_T})^{m-1} \| u - v \|_{X_T}. \]
Choose \( \gamma = 2C\| f \|_{W^{s,2}} \) and \( 0 < T < 1 \) so that
\[ T < \left( \frac{1}{C_0(1 + \| f \|_{W^{s,2}})^{m-1}} \right)^{1/r}, \]
where \( C_0 \) is a constant. Then it follows that \( \Phi \) maps \( B \) into \( B \) and is a contraction mapping on \( B \). This proves the theorem. \( \square \)

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References


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