ON ISOLATED SINGULARITIES OF MINIMAL SURFACES

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We here want to give a brief discussion of some questions related to minimal surfaces with isolated singularities; the many questions related to minimal surfaces with more complicated singular sets are not considered.

We first make our terminology precise. For simplicity of exposition we discuss embedded submanifolds of Euclidean space - making the necessary comments about the more general Riemannian setting at appropriate points.

M will denote a smooth n-dimensional embedded submanifold of \( \mathbb{R}^{n+k} \), \( n \geq 2 \), \( k \geq 1 \), where we always use the term "embedded" to mean locally properly embedded. Thus for each \( y \in M \) there is an open ball \( B_\rho(y) \) with centre \( y \) and radius \( \rho > 0 \), and a \( C^2 \) diffeomorphism \( \psi \) of \( B_\rho(y) \) onto \( B_\rho(0) \) such that \( \psi(M \cap B_\rho(y)) = \mathbb{R}^n \cap B_\rho(0) \). Here and subsequently we identify \( \mathbb{R}^n \) with the subspace of \( \mathbb{R}^{n+k} \) consisting of all points \( (x^1, \ldots, x^{n+k}) \) such that \( x^j = 0 \) \( \forall j = n+1, \ldots, n+k \).

\( M \) is said to be minimal if the mean curvature of \( M \) is identically zero. As is well-known, this condition is equivalent to the local area minimising property: for each \( y \in M \) there is some open ball \( B_\rho(y) \subset \mathbb{R}^{n+k} \) such that
\[ H^n(M \cap B_p(y)) \leq H^n(\tilde{M} \cap B_p(y)) \]  
(H^n = k-dimensional Hausdorff measure)

whenever \( \tilde{M} = \varphi(M) \), where \( \varphi : B_p(y) \rightarrow B_p(y) \) is a \( C^1 \) diffeomorphism such that \( \{ x \in B_p(y) : \varphi(x) \neq x \} \) is contained in a compact subset of \( B_p(y) \). This explains the use of the term minimal.

The regular set \( \text{reg} \ M \) of \( M \) is defined to be the set of all points \( y \in \text{clos} \ M \) (closure of \( M \) taken in \( \mathbb{R}^{n+k} \)) such that \( B_p(y) \cap \text{clos} \ M \) is an embedded \( n \)-dimensional \( C^2 \) submanifold of \( \mathbb{R}^{n+k} \) for some \( p > 0 \). The singular set \( \text{sing} \ M \) of \( M \) is defined by \( \text{sing} \ M = \text{clos} \ M \setminus \text{reg} \ M \). By definition \( \text{sing} \ M \) is a closed subset of \( \mathbb{R}^{n+k} \) and \( \text{reg} \ M \supset M \); if the inclusion \( \text{reg} \ M \supset M \) is strict, then there are removable singularities. We always assume here that such singularities have been removed, so that

\[ M = \text{reg} \ M, \quad \text{sing} \ M = \text{clos} \ M \setminus M. \]

A point \( y \in \mathbb{R}^{n+k} \) is said to be an isolated singularity of \( M \) if \( y \in \text{sing} \ M \) and \( \text{sing} \ M \cap B_p(y) = \{ y \} \) for some \( p > 0 \).

The simplest examples of embedded minimal \( M \) with isolated singularities (indeed the only known codimension 1 examples until the work [CHS] to be described in §2 below) are the embedded minimal cones; that is, minimal \( M \) representable in the form

(0.1) \[ M = \{ \lambda y : y \in \Sigma, \ 0 < \lambda < \infty \}, \]

where \( \Sigma \) is a compact embedded \((n-1)\)-dimensional submanifold of \( S^{n+k-1} \).
There are many such minimal cones (see e.g. [HL]).

A simple example (corresponding to \( n = 3 \), \( k = 1 \)) is

\[
M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \sim \{0\} : (x^1)^2 + (x^2)^2 = (x^3)^2 + (x^4)^2 \}.
\]

In this case the corresponding submanifold \( \Sigma \subset S^3 \) (as in (0.1)) is just the two dimensional flat torus

\[
\left( \frac{1}{\sqrt{2}} \mathbb{S}^1 \right) \times \left( \frac{1}{\sqrt{2}} \mathbb{S}^1 \right) \subset \mathbb{S}^3 \subset \mathbb{R}^4 \equiv \mathbb{R}^2 \times \mathbb{R}^2.
\]

More generally for any \( p \geq 1 \)

\[
M = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^p \sim \{0\} : |x|^2 = |y|^2 \}
\]

is a \((2p-1)\)-dimensional minimal cone, corresponding to \( \Sigma \subset S^{2p-1} \) defined by

\[
\Sigma = \left( \frac{1}{\sqrt{2}} \mathbb{S}^{p-1} \right) \times \left( \frac{1}{\sqrt{2}} \mathbb{S}^{p-1} \right).
\]

The study of minimal surfaces generally is closely related to quasilinear P.D.E. theory, by virtue of the fact that \( U \) is an open subset of \( \mathbb{R}^n \) and if \( u = (u^1, \ldots, u^k) : U \to \mathbb{R}^k \) is a \( C^2(U) \) function with values in \( \mathbb{R}^k \), then

\[
(0.2) \quad M = \text{graph } u = \{(x, u(x)) : x \in U\}
\]

is minimal if and only if \( u \) satisfies the system of equations

\[
(0.3) \quad \sum_{i,j=1}^{n} g^{ij}_{\ell} D_{i,j}u^\ell = 0 \quad \text{on } U, \quad \ell = 1, \ldots, k.
\]

Here \( (g^{ij}_{\ell}) = (g^{ij}_{\ell})^{-1} \), \( g^{ij}_{\ell} = \delta^{ij} + D_i u \cdot D_j u \). The system (0.3) is called the minimal surface system (minimal surface equation in
case $k = 1$), and with respect to suitably chosen coordinate axes any minimal $M \subset \mathbb{R}^{n+k}$ can be represented locally as graph $u$ for some $u$ satisfying (0.3). All this is readily checked by directly computing the mean curvature vector of $M$ in case $M$ has the form (0.2). (See, for example, [OJ.)

We should make a point here: it is not in general true that $M$ can be represented in the form (0.2) in the neighbourhood of an isolated singular point. Indeed in the codimension 1 case (when $k = 1$), no $M$ of the form (0.2) can have isolated singular points. For this and related results see [FR], [B], [MM], [SL2, 3]. On the other hand in case $n = 4$, $k = 3$ there is an example due to Lawson and Osserman [LO] of an $M$ having the form (0.2) and having an isolated singularity. The example is also of the form (0.1) for some compact smooth $\Sigma \subset S^3$.

The outline of the present article is as follows: in §1 we discuss an old question (still unsettled) as to whether an embedded minimal surface in $\mathbb{R}^3$ can have an isolated singularity. §2 gives a brief summary of recent work of Caffarelli, Hardt, Simon [CHS] on examples of minimal submanifolds with isolated singularities, obtained by perturbation of minimal cones (as in (0.1)). In §3 we briefly discuss the important question of whether or not a minimal surface is asymptotic to a minimal cone near an isolated singular point.

For reference in these subsequent sections we here make some remarks concerning the area growth of minimal surfaces near an isolated singularity. Thus we let $M \subset \mathbb{R}^{n+k}$ be as above and
we suppose that sing $\mathcal{M} \cap B_{\rho_0} = \{0\}$ for some $\rho_0 > 0$. Here and subsequently we abbreviate $B_\rho = B_\rho(0)$. Since we assume $\mathcal{M}$ is locally properly embedded we have

$$(0.4) \quad H^2(\mathcal{M} \setminus (B_\rho \sim B_\sigma)) < \infty, \quad 0 < \sigma < \rho < \rho_0.$$ 

Now by plugging in a function of the form $\psi(|x|)x$ in the first variation formula (see [AW] or [SL1] or [MS] for a discussion) we get the identity

$$(0.5) \quad \int_M \left[ n \psi(|x|) + |x| \psi'(|x|) \nabla |x|^2 \right] = 0,$$ 

provided support $\psi$ is a compact subset of $(0, \rho_0)$. In (0.5), $\nabla$ denotes gradient on $\mathcal{M}$; that is $\nabla f(x)$ is the orthogonal projection of the ordinary $\mathbb{R}^{n+k}$ gradient of a function $f$ onto the tangent space $\mathbb{T}_x \mathcal{M}$.

We now choose $\rho \in (0, \rho_0), \sigma \in (0, \rho)$ and replace $\psi$ by the function $\psi_{\sigma, \rho}$ defined by

$$\psi_{\sigma, \rho}(t) = \phi_\sigma(t) \psi(t/\rho),$$

where $\phi'_\sigma(t) \geq 0 \ \forall t$, $\phi_\sigma(t) \equiv 0$ for $t < \sigma/2$, $\psi(t) \equiv 1$ for $t > \sigma$, $\psi'(t) \leq 0 \ \forall t$, and $\psi(t) = 0$ for $t \geq 1$. Then (0.5) implies, for $0 < \sigma < \rho < \rho_0$, that

$$(0.6) \quad n \int_M \phi_\sigma(|x|) \psi(|x|/\rho) - \rho \frac{d}{d \rho} \int_M \phi_\sigma(|x|) \psi(|x|/\rho) \nabla |x|^2 \leq 0,$$
where we have used \( \frac{\partial}{\partial \rho} [\psi(t/\rho)] = \frac{\partial}{\partial t} [\psi(t/\rho)] \) (the common value being \( t/\rho \psi(t/\rho) \)). Taking \( \varepsilon \in (0, 1) \), \( \psi \equiv 1 \) on \([0, 1-\varepsilon] \) and multiplying through by \( \rho^{-n-1} \) and rearranging the resulting expression, making use of the fact that \( \rho^{-n} |\psi'(|x|/\rho)| \geq \frac{(1-\varepsilon)^n}{|x|^n} |\psi'(|x|/\rho)| \) (because \( \psi(t/\rho) \equiv 1 \) for \( t \leq (1-\varepsilon)\rho \)), we then deduce

\[
- \frac{d}{d \rho} \left[ \int_M \phi_\sigma(|x|) \frac{1-|x|^n}{(1-\varepsilon)^n |x|^n} \psi(|x|/\rho) \right] + \frac{d}{d \rho} \left[ \int_M \phi_\sigma(|x|) \psi(|x|/\rho) \right] \geq 0 ,
\]

so that in particular

\[
(0.7) \quad \int_M \phi_\sigma(|x|) \frac{1-|x|^n}{(1-\varepsilon)^n |x|^n} \left( \psi(|x|/\rho) - \psi(|x|/\tau) \right) + \tau^{-n} \int_M \phi_\sigma(|x|) \psi(|x|/\tau)
\]

\[
\leq \rho^{-n} \int_M \phi_\sigma(|x|) \psi(|x|/\rho)
\]

\[
\leq - n^{-1} \rho^{-n} \int_M \phi_\sigma(|x|) \psi'(|x|/\rho)
\]

where, in the last line, we used (0.6) again. This is valid for \( \sigma < \tau < \rho < \rho_0 \). Holding \( \rho, \tau \) fixed, and letting \( \sigma \to 0 \) we deduce by virtue of (0.4) that

\[
(0.8) \quad H^n(M \cap B_\tau) < \infty \quad \forall \tau < \rho_0 .
\]

Then we can let \( \phi_\sigma + 1 \) and \( \psi + \chi \), where \( \chi \) is the characteristic function of the interval \([0, 1] \). This gives

\[
(0.9) \quad \int_{M \cap (B_{\rho - \tau} \setminus B_\tau)} |x|^{-n} \left( 1 - |\nabla |x|^2 \right) + \tau^{-n} H^n(M \cap B_\tau)
\]

\[
\leq \rho^{-n} H^n(M \cap B_\rho) < \infty , \quad 0 < \tau < \rho < \rho_0 .
\]
This is the well-known \textit{monotonicity formula} for minimal surfaces, which we have thus shown to be valid if \( M \) has an isolated singularity at \( 0 \). (Actually it is valid in the presence of much more serious singularities - in fact for arbitrary stationary varifolds - but in this case one needs \textit{a-priori} to assume finiteness of the area, which we did \textit{not} need to assume here.)

A formula like (0.9) continues to hold in the more general case when \( M \) is a submanifold of a Riemannian manifold \( N \); in this case the \( B_\rho \) are Geodesic balls in \( N \), \( \rho \) must be sufficiently small, and there is an additional factor of the form \((1+\rho \phi)\) on the right hand side (\( \phi \) a constant depending on \( N \)).

\section{2-dimensional surfaces in \( \mathbb{R}^3 \)}

Here we suppose \( n = 2 \), \( k = 1 \); thus \( M \) is an embedded 2-dimensional minimal surface in \( \mathbb{R}^3 \). The question is whether or not such surfaces can have an isolated singularity at \( 0 \).

We shall here sketch the simple proof that such an isolated singularity \textit{cannot} exist if we make the additional assumption that \( M \) is stable in \( B_{\rho_0} \sim \{0\} \) for some \( \rho_0 > 0 \). Thus we shall assume that \( \text{sing} \ M \cap \{ B_{\rho_0} \sim \{0\} \} = \emptyset \) and

\begin{equation}
\int_{M \cap B_{\rho_0} \sim \{0\}} |A|^2 \phi^2 \leq \int_{M \cap B_{\rho_0} \sim \{0\}} |\nabla \phi|^2 \tag{1.1}
\end{equation}

whenever \( \phi \in B_{\rho_0} \sim \{0\} \) is compact, where \( A \) denotes the
second fundamental form of $M$. ((1.1) is just the stability inequality; see for example [Sll] for an elementary discussion.)

We want to prove $0 \in \text{reg } M$. Note first that (1.1), taken together with the area bounds (0.9), implies immediately that

\[(1.2) \quad \int_{M \cap B_{\rho}} |A|^2 < \infty \quad \forall \rho \in (0, \rho_0).\]

(To check this, just take $\varphi$ in (1.1) with $\varphi \equiv 1$ on $B_{\rho} \sim B_{\rho_0}$, $\varphi = 0$ on $\partial B_{\rho_0}$, $\varphi \equiv 0$ in $B_{\rho/2}$, and $|\nabla \varphi| \leq c/\sigma$, then let $\sigma \to 0$ and use (0.9) to bound the right hand side.) Also by the curvature estimates ([SSY] of [Sll]) for stable minimal surfaces we have that

\[(1.3) \quad |A(x)| \leq c/|x|, \quad x \in M \cap B_{\rho_0/2}.\]

Now let $\rho_k \to 0$ and let $M_k = \left\{ \rho_k^{-1}x : x \in M \right\}$. By (1.3) and the Arzela-Ascoli lemma we can select a subsequence \(k' \subset k\) such that \(\left\{ M_{k'} \right\}\) converges (in a $C^1$ sense) to a minimal surface $C$ with $\text{sing } C = \{0\}$. (Notice that we do this by locally representing the $M_{k'}$ as graphs of functions satisfying the minimal surface equation, which is possible in a uniform way by (1.3), because (1.3) tells us that the second fundamental form $A_{k'}$ of $M_{k'}$ is uniformly bounded in any annular region $B_{\rho} \sim B_{\rho_0}$ as $k' \to \infty$.) By virtue of the monotonicity formula (0.9), we easily check that $\int_{C \cap B_{\rho}} (1 - |\nabla |x|)^2 = 0$ ($\nabla = \text{gradient operator on } C$, so
that $|\nabla x| \equiv 1$ on $C$). Then $C$ must be a cone with vertex at the origin, and $C \cap S^2$ is a 1-dimensional embedded minimal surface (i.e. a geodesic) in $S^2$, and hence must be a great circle in $S^2$. Thus we finally deduce that $C$ is a plane through the origin.

Without loss of generality, let us suppose that the unit normal of $C$ is $e_3 = (0, 0, 1)$.

Since $M_k$ approaches $C$ locally (away from 0) in a $C^1$ fashion, we see that

$$\sup_{\Sigma_{k,r}} \left( B_k - B_{k'} |\rho_{k'} \rho_{k'}/2 \right) (1 - |\nabla e_3|) \to 0 \text{ as } k' \to \infty,$$

where $\nu$ = unit normal of $M$. In particular if we take any $r > k$ and if we let $\Sigma_{k,r}$ be any component of $M \cap \left( B_k - B_{k'} \rho_{k'} \rho_{k'}/2 \right)$, and if we write $L_{k,r} = \partial \Sigma_{k,r}$, then we see that the Gauss map $\nu : M \to S^2$ takes $L_{k,r}$ to an $\epsilon$-neighbourhood of $e_3$, where $\epsilon \to 0$ as $k, r \to \infty$.

On the other hand the Jacobian (area magnification factor) of the Gauss map is minus the Gauss curvature of $M$, which is $\frac{1}{2} |A|^2$. Hence the area of the Gauss image of $\Sigma_{k,r}$ is $\frac{1}{2} \int_{\Sigma_{k,r}} |A|^2$, which converges to zero as $k \to \infty$ by (1.2), and furthermore the Gauss map is either a constant map on an open map (because the Gauss curvature vanishes at only isolated points of $\Sigma_{k,r}$ unless $\Sigma_{k,r}$ is contained in a plane). We therefore deduce that the Gauss image of $\Sigma_{k,r}$ must be contained in the same $\epsilon$-neighbourhood of $e_3$ as $L_{k,r}$, where $\epsilon \to 0$ as $k \to \infty$. 
Letting $r \to \infty$, holding $k$ fixed (but large), we can thus deduce that the oscillation of $\nu$ on any component of $B \cap M$ is small, $B = B_{\rho_k'}$. Writing $B' = B_{\rho_k'/2}$, it follows that for each component $M^\#_r$ of $B \cap M$ with $0 \in \text{clos} \, M^\#_r$ we have a minimal graph $G$ with

$$0 \in (\text{clos} \, G) \cap B' = (\text{clos} \, M^\#_r) \cap B',$$

provided $k'$ is sufficiently large. (In checking this one needs to use the fact that the oscillation of $\nu$ on $M^\#_r$ is small, together with the fact that $M^\#_r$ is connected, embedded, and $\partial M^\#_r \subset \partial B$.) But by the discussion of the introduction, we know that isolated singularities of codimension 1 minimal graphs are removable, hence we have

$$\text{clos} \, M^\#_r \cap B' = G,$$

where $G = \text{graph} \, u$, with $u(0) = 0$ and with $u$ satisfying the minimal surface equation in some neighbourhood of the origin in $\mathbb{R}^2$.

This essentially completes the proof, except that we have to check that there cannot be more than one component $M^\#_r$ of $M \cap B$ whose closure contains the origin. However if there were another, then we could apply the above argument to it in order to deduce that there are two minimal graphs $G_1, G_2$ with $G_1 \cap G_2 = \{0\}$. Since the difference of two solutions of the minimal surface equation satisfies the maximum principle, this is impossible.

Notice that the above result has been extended to the case $2 < n \leq 6$; thus if $2 < n \leq 6$ ($k = 1$ and $M$ stable in
B_0 \sim \{0\} still being assumed), then M cannot have an isolated singularity at 0. Indeed if $\mu^n(M \cap B_0) < \infty$ (which is automatic from (0.9) in case M has an isolated singularity at 0 and $\rho_0$ is sufficiently small) and if M is stable in $B_0 \sim \text{sing } M$, then

\begin{equation}
\mu^{n-2}(\text{sing } M \cap B_0) = 0 \Rightarrow \text{sing } M \cap B_0 = \emptyset.
\end{equation}

(In particular this guarantees isolated singularities are removable for $2 < n \leq 6$.) (1.4) is proved in the recent work of Schoen and Simon [SS1]. The results of [SS1] are actually only stated for the case when M is stable in $B_0$, but the reader will see that the proofs of [SS1] actually apply to the case when M is only stable in $B_0 \sim \text{sing } M$ (in the sense that (1.1) holds for any $\varphi \in C_0^1(B_0)$).

All this generalizes to the case when M is a codimension 1 embedded submanifold of a general Riemannian manifold, and to the case when M has bounded (rather than zero) mean curvature. In fact [SS1] is already presented in this setting, and the main change needed in the above 2-dimensional argument relates to the fact that the Gauss map is no longer necessarily open. One then needs to use the iterative procedure described in §2 of [SS2] to prove that the oscillation of $\varphi$ on $M^k$ is small as before.
§2. Examples of minimal surfaces with isolated singularities

The material in this section is an outline of some parts of recent joint work [CHS].

Let \( E \) be any smooth \((n-1)\)-dimensional submanifold of \( S^{n+k-1} \), \( 1 \leq k \leq n \geq 2 \), and let \( C \) be the cone over \( E \). Thus

\[
C = \{ \lambda \omega : \lambda > 0, \omega \in E \}
\]

and \( C \) has an isolated singularity at \( 0 \) unless it is a \( n \)-dimensional plane through the origin. Write also

\[
C_r = C \cap B_r,
\]

so that

\[
\partial C_1 = \Sigma.
\]

For the moment we are not assuming \( C \) is minimal.

We consider first the linear Dirichlet problem

\[
\begin{cases}
L_C u = f \text{ on } C_1 \\
u = \psi \text{ on } \Sigma,
\end{cases}
\]

where \( f, \psi \) are given functions on \( C_1 \) and \( \Sigma \) respectively, and

where \( L_C \) is a linear-second order operator of the form

\[
L_C u = \Delta_C u + Qu,
\]

where \( \Delta_C \) denotes the Laplace-Beltrami operator for \( C \), and \( Q \) has
the form

\[ Q(x) = r^{-2}q(\omega), \; x \in C, \; r = |x|, \; \omega = x/|x| \in \Sigma. \]

Since \( C \) is a cone, we have

\[ L_C u = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial u}{\partial r} \right) + r^{-2} \left( \Delta_{\Sigma} u + q(\omega) u \right) \tag{2.2} \]

where on the right \( u = u(r, \omega), \; 0 < r < 1, \; \omega \in \Sigma \) and \( \Delta_{\Sigma} \) (the Laplace-Beltrami operator for \( \Sigma \)) acts on \( u(r, \omega) \) as a function of \( \omega \in \Sigma \) with \( r \) fixed.

Our initial aim in this section is to discuss the following simple question concerning solutions of (2.1):

Suppose \( p > 0 \),

\[ \|f\|_p = c \; r^{p-2} \tag{2.3}, \quad \forall 0 < r < 1, \]

where \( \|f\|_p = \left( \int_{\Sigma} f^2(r\omega) d\omega \right)^{\frac{1}{2}} \); then for which boundary data \( \psi \) in (2.1) can we find a solution \( u \) of (2.1) such that \( u \) decays near 0 in the sense

\[ \|u\|_p \leq c' \; r^{p} \tag{2.4}, \quad \forall 0 < r < 1. \]

(Notice that the requirement \( \|f\|_p \leq c \; r^{p-2} \) is evidently necessary for there to exist any such boundary data \( \psi \) in view of the form of the operator \( L_C \).)

We write \( L_C = \Delta_{\Sigma} + q \) and we let

\[ \mu_1 \leq \mu_2 \leq \ldots \quad (\mu_k \to \infty \text{ as } k \to \infty) \]
denote the eigenvalues of $L_\Sigma$, and let

$$\phi_1, \phi_2, \ldots$$

be a corresponding orthonormal basis for $L^2(\Sigma)$. (Notice that these $\phi_j$ are automatically of class $C^{2,\alpha}$ on $\Sigma$ by virtue of the assumption that $q$ is $C^{0,\alpha}$.) Thus any $u \in L^2(\Sigma)$ can be written

$$u(x) = u(r\omega) = \sum_{j=1}^{\infty} a_j(r)\phi_j(\omega)$$

where $\sum_{j=1}^{\infty} a_j(r) < \infty$ a.e. $r \in [0, 1]$, and (by (2.2)) such a $u$ is a solution of the equation $L_\Sigma u = f$ (in the generalized sense) if and only if

$$(2.5) \quad r^2a''_j + (n-1)ra'_j - \mu_ja_j(r) = r^2f_j(r), \quad j = 1, 2, \ldots$$

(in the generalized sense), where

$$f_j(r) = \int_\Sigma f(r\omega)\phi_j(\omega) \, d\omega \equiv \langle f(r\omega), \phi_j(\omega) \rangle_{L^2(\Sigma)}$$

where $0 < r < 1$.

The homogeneous equation (i.e. (2.5) with $f_j = 0$) has solutions $cr^\gamma$, where $\gamma$ is any root of

$$(2.5)' \quad \gamma^2 + (n-2)\gamma - \mu_j = 0,$$

so
\[ \gamma = -\frac{n-2}{2} \pm \sqrt{(\frac{n-2}{2})^2 + \mu_j} \].

For each \( j \) we let \( \gamma_j \) be the root corresponding to the plus sign.

Then since \( w_j = r^{-\gamma_j} a_j \) satisfies the equation

\[ \left\{ r^{n-1+2\gamma_j} \Phi^j \right\} = r^{n-1+\gamma_j} f_j \] we see that there are solutions

\[ (2.6) \quad a_j(r) = \Re \left[ \alpha_j r^{\gamma_j} + \int_0^r \int_0^\tau s^{-(n-1)-2\gamma_j} f_j(\tau) d\tau ds \right] + \int_0^r \int_0^\tau \beta_j r^{\gamma_j} f_j(\tau) d\tau ds, \]

where \( \alpha_j, \beta_j \) are constants with \( \beta_j > 0 \). We now make the additional assumption on \( p \) that there is an integer \( j_0 \geq 1 \) such that \( p > 0 \) (as before) and

\[ (2.7) \quad \Re \{ Y_{j_0} \} < p < \Re \{ Y_{j_0+1} \}. \]

The expressions for \( a_j \) in (2.6) make sense, and give a solution of (2.5), provided that we take

\[ \begin{cases} \alpha_j \in \mathbb{R} \text{ arbitrary, and } \beta_j = 1 \text{ for } j \geq j_0 + 1 \\ \alpha_j = 0, \text{ and } \beta_j = 0 \text{ for } j \leq j_0. \end{cases} \]

Indeed one easily checks that then

\[ u = \sum_{j=1}^{\infty} a_j(r) \varphi_j(w) \]

satisfies (2.1) with
\[
\psi = \sum_{j=j_0+1}^{j_0} \alpha_j \phi_j + \sum_{j=1}^{j_0} \text{Re} \left( \int_0^s \left. \int_0^t f_j(\tau) \, d\tau \, ds \right) \right) -(n-1)-2\gamma_j \int_0^s \tau^{n-1+\gamma_j} f_j(\tau) \, d\tau \, ds \right)
\]

and that

\[
(2.8) \quad \|u\|_r \leq c \left( \sum_{j=j_0+1}^{\infty} \alpha_j^2 + \sup_{0 < t < 1} \tau^{2-p} \|f\|_r \right) r^p, \quad 0 < r < 1
\]

as required. Furthermore one readily checks that \( u \) is the unique solution of (2.1) satisfying \( \sup_{0 < t < 1} r^{-p} \|u\|_r < \infty \), together with the boundary data

\[
(2.9) \quad (\lim_{r \uparrow 1} u(\cdot, \omega), \varphi_j(\omega))_{L^2(\Sigma)} = \alpha_j, \quad j \geq j_0 + 1.
\]

(We want to emphasize particularly that we have no control over the numbers \( (\lim_{r \uparrow 1} u(\cdot, \omega), \varphi_j(\omega))_{L^2(\Sigma)} \), \( j \leq j_0 \); these are uniquely determined by \( f \).)

Using the notation \( \Pi_j : L^2(\Sigma) + L^2(\Sigma) \) to denote the operator \( w \mapsto \sum_{j=j_0+1}^{\infty} \left( w, \varphi_j \right)_{L^2(\Sigma)} \varphi_j \), we have thus proved that for any given \( \psi \in L^2(\Sigma) \) there is a unique solution \( u \) of

\[
(2.10) \quad \begin{cases}
L_c u = f \quad \text{on} \quad \Sigma_c \\
\Pi_j (u-\psi) = 0 \quad \text{on} \quad \partial \Sigma_c \\
\end{cases}
\]

with \( \sup_{0 < r < 1} r^{-p} \|u\|_r < \infty \); this solution satisfies
\[ (2.10) \quad \| u \|_r \leq c \left( \| \int_0^r \psi \|_{L^2(E)} + \sup_{0 < t < r} t^{2-p} \| f \|_r \right)^{r^p}, \]

\[ \| (1-\int_0^r) u \|_{L^2(E)} \leq c \sup_{0 < t < r} t^{2-p} \| f \|_r, \]

where \( u|_E(\omega) = \lim_{r \to 1} u(\omega r) \). Of course we are still assuming (2.3) and (2.7).

To describe applications of this to non-linear equations (and to minimal surfaces) consider the Banach spaces

\[ B_p = \left\{ u \in C^2(E) : \| u \|_{B_p} \right\}, \]

where

\[ \| u \|_{B_p} = \sup_{0 < r < 1} r^{-p} \left( \sum_{j=0}^{2} r^j \| \nabla^j u \|_{0; r} + r^{2+\alpha} [\nabla^2 u]_{\alpha; r} \right) < \infty, \]

where we use the notation that \( \nabla^2 u \) denotes the covariant Hessian of \( u \),

\[ |w|_{0; r} = \sup_{0 < r < 1} |w|, \quad [w]_{\alpha; r} = \sup_{x, y \in C \sim r/2} \frac{|w(x) - w(y)|}{|x-y|^\alpha}, \]

and where \( \alpha \) is a fixed constant in the interval \((0, 1)\).

Now consider the quasilinear problem of finding \( u \in B_p \) such that

\[ (2.11) \]

\[
\begin{align*}
Lu + N(u) &= f \quad \text{on} \quad C_1 \\
\Pi_0 (u-\psi) &= 0 \quad \text{on} \quad \Sigma,
\end{align*}
\]
where $f$, $\psi$, $j_0$ are as above but where we now make the additional assumption

$$\text{Re } j_0 \geq 1,$$

so that (2.7) implies $p > 1$. Concerning the operator $N$ we assume

$$Nu = N^{(1)}(x, u/r, Vu) \cdot \nabla^2 u + r^{-1} N^{(2)}(x, u/r, Vu).$$

Here, for $(x, z, p) \in C^1 \times \mathbb{R} \times \mathbb{R}^{n+k}$, $N^{(1)}(x, z, p)$ is a symmetric bilinear form on $\mathbb{R}^{n+k}$ and $N^{(1)}(x, u/r, Vu) \cdot \nabla^2 u$ should be taken to mean trace $S_1 \circ S_2$, where $S_1, S_2$ are the symmetric bilinear transformations on $T_x C \times T_x C$ associated with $N^{(1)}(x, u/r, Vu)|_{T_x C \times T_x C}$ and $\nabla^2 u(x)$. We also assume that $N^{(1)}, N^{(2)}$ have a $C^2$ dependence on $(x, z, p) \neq 0$ and satisfy the structural conditions

$$\left\{ \begin{array}{l}
\sum_{j=1}^{2} \left[ |N^{(1)}_{x}(x, z, p)| + |N^{(2)}_{x}(x, z, p)| \right] + |N^{(1)}_{z}(x, z, p)| \\
+ |N^{(1)}_{p}(x, z, p)| + r \sum_{j=1}^{2} \left( |z| + |p| \right) |N^{(j)}_{xz}(x, z, p) + |N^{(j)}_{zp}(x, z, p)| \right] \leq \mu \left( |z| + |p| \right) \\
+ \sum_{j=1}^{2} \left[ |N^{(j)}_{zz}(x, z, p)| + |N^{(j)}_{zp}(x, z, p)| + |N^{(j)}_{pp}(x, z, p)| \right] \leq \mu \left( |z| + |p| \right) \\
\end{array} \right\}$$

$$|N^{(1)}(x, z, p)| + |N^{(2)}(x, z, p)| + |N^{(2)}_{p}(x, z, p)| \leq \mu (|z| + |p|)^2.$$
Subject to these assumptions, the result described in (2.10) above, together with a standard application of the implicit function theorem (or the contraction mapping principle), enables us to assert that if (2.3), (2.7), (2.11), (2.12), (2.14) hold, then there is a constant $\beta_0 > 0$ such that the problem

$$
\begin{cases}
Lu + N(u) = \beta f & \text{on } C_1 \\
\Pi_j (u - \beta \psi) = 0 & \text{on } \Sigma
\end{cases}
$$

(2.15)

is solvable for any $\beta \leq \beta_0$, with $u \in B_p$ satisfying

$$
\|u - u_1\|_{B_p} \leq c \beta^2,
$$

where $u_1$ is the solution of the linear problem (2.10) with $\beta f, \beta \psi$ in place of $f, \psi$.

If $C_1$ is minimal of codimension 1 (i.e. $k = 1$), and if $M$ is the graph of a function $u \in B_p$ over $C_1$:

$$
M = \{ x + v(x)u(x) : x \in C_1 \} \quad (v = \text{unit normal of } C_1),
$$

then $M$ is a minimal hypersurface if and only if $u$ satisfies

$$
M_C u = 0 \text{ on } C_1, \text{ where } M_C \text{ is the minimal surface operator (i.e. the mean curvature operator). As is well known, } M_C \text{ has the form}
$$

$$
M_C(u) = L_C u + N(u),$$
with \( N \) as in (2.13), (2.14) and with \( L_C \) as in (2.2) with

\[
q = |A|^2 \quad \text{on} \quad \Sigma,
\]

\( A = \) second fundamental form of \( C \). Therefore the above general existence result gives a large new class of examples of minimal surfaces with isolated singularities. Specifically:

**Theorem 1.** Suppose \( C \) is minimal, \( k = 1 \), \( q \) is as in (2.16), and suppose \( \gamma_0 \) is chosen so that (2.7), (2.12) hold for some \( p > 1 \). Then for any given \( \psi \in C^{2,\alpha}(\Sigma) \) with \( (1-\Pi_{\gamma})\psi = 0 \), there is a 1-parameter family \( \{u_\beta\}_{0 < \beta \leq \beta_0} \subset B_p \) of solutions of

\[
M_C u = 0 \quad \text{such that}
\]

\[
\lim_{\beta \to 0} \beta^{-1} u_\beta(x) = \psi, \quad x \in C_1 \cup \Sigma.
\]

In fact we have

\[
\dot{u}_0 = \Re \sum_{j \geq j_0 + 1} \gamma_j (\psi, \phi_j) L_2(\Sigma) \phi_j \quad \text{on} \quad C_1.
\]

All the above discussion extends straightforwardly to the case when our functions are sections of the normal bundle \( TC^\perp \) of \( C \), provided we take

\[
L_C = \Delta_C^\perp + r^{-2} q(\omega),
\]

where \( \Delta_C^\perp \) is the normal Laplacian (see [SJ]) and \( q(\omega) : T_{\omega}^\perp \to T_{\omega}^\perp \).
is linear with smooth dependence on \( \omega \). If \( C \) is minimal, the minimal surface operator \( M_C \) in this arbitrary codimension case is an operator taking smooth sections of \( T_C \) to smooth sections of \( T_C \) with the form

\[
M_C = L_C + N,
\]

with \( L_C \) as in (2.17) with

\[
(2.18) \quad q(\omega) = A_\omega
\]

where \( A_\omega : (T^\perp_\omega) \to (T^\perp_\omega) \) is defined by

\[
v_2 \cdot A_\omega (v_1) = \text{trace} (v_1 \cdot A_\omega) \circ (v_2 \cdot A_\omega),
\]

where \( A \) is the second fundamental form of \( \Sigma \) (thus \( v_1 \cdot A_\omega \) is a bilinear form on \( T_\omega \Sigma \times T_\omega \Sigma \)), and where \( N \) satisfies similar structural conditions to the \( N \) in (2.13), (2.14). (As before, \( M_C u = 0 \) on \( C_1 \) if and only if graph \( u = \{x + u(x) : x \in C_1\} \) is an \( n \)-dimensional minimal submanifold of \( \mathbb{R}^{n+k} \).

We thus have in particular the following theorem.

**Theorem 2.** Suppose \( C \) is minimal, \( k \geq 1 \), \( q \) is as in (2.18) and suppose \( j_0 \) is chosen so that (2.7), (2.12) hold for some \( p > 1 \). Then for any given \( \psi \in C^2(\Sigma, T^\perp \Sigma) \) there is a 1-parameter family \( \{u_B\}_{0 \leq B \leq B_0} \subset B^\perp_p \) of solutions of \( M_C u = 0 \) such that \( u_0 |_{\Sigma} = \psi \).
Here the notation is as follows: \( \mathfrak{u}_0 = \lim_{\beta \to 0} \beta^{-1} \mathfrak{u}_\beta \) as before, \( \mathfrak{B}_p \)
denotes the set of \( C^2, \alpha(\mathfrak{c}, \mathbb{R}^{n+k}) \) functions \( w = (w^1, \ldots, w^{n+k}) \) such that each component \( w^j \) is in \( \mathfrak{B}_p \) and such that
\( w(x) \in T_x \mathfrak{c} \quad \forall x \in \mathfrak{c} \), and \( C^{2, \alpha}(\Sigma, T\mathfrak{c}^\perp) \) denotes the set of
\( C^{2, \alpha}(\Sigma, \mathbb{R}^{n+k}) \) functions \( w = (w^1, \ldots, w^{n+k}) \) such that each component \( w^j \) is in \( C^{2, \alpha}(\Sigma) \) and \( w(x) \in T_x \Sigma \quad \forall x \in \Sigma \).

The question now arises whether or not these perturbed solutions are stable in case the underlying minimal cone is stable.

To facilitate further discussion we first derive a criterion (involving eigenvalues of \( L_\Sigma \)) for the cone \( \mathfrak{c} \) to be stable.

Since there are no 2-dimensional minimal cones other than the 2 planes, we assume \( n \geq 3 \) for the remainder of this section.

To begin, let us suppose \( \mathfrak{c} \) is an arbitrary cone (not necessarily minimal) and let \( L_\mathfrak{c} = \Delta_\mathfrak{c} + Q \) as before.

Let \( \zeta \in C^2_0(\mathfrak{c}, \mathfrak{c}) \) be arbitrary, and write
\[
\zeta(rw) = \sum_{j=1}^{\infty} a_j(r) \varphi_j(\omega), \quad \text{where} \quad a_j \in C^1_0((0, 1)).
\]

Then by (2.2)
\[
\int_{\mathfrak{c}} L_\mathfrak{c} \zeta = \int_0^1 \left[ \sum_{j=1}^{\infty} \frac{\partial}{\partial r}(r^{n-1}a'_j) \varphi_j(\omega) + r^{n-3}a_j(L_\mathfrak{c} \varphi_j) \right] \left( \sum_{k=1}^{\infty} a_k \varphi_k \right) dw \ dr,
\]
so that since \( L_\mathfrak{c} \varphi_j = -\mu_j \varphi_j \) and since \( \varphi_j \) are orthonormal in \( L^2(\Sigma) \), we get
\[
(2.19) \quad \int_{\mathfrak{c}} L_\mathfrak{c} \zeta = -\int_0^1 \left[ r^{n-1} \sum_{j=1}^{\infty} (a'_j)^2 + r^{n-3} \sum_{j=1}^{\infty} \mu_j a_j^2 \right] dr.
\]
Now using integration by parts we have

\begin{equation}
(2.20) \quad \int_0^1 r^{n-3} \sum_{j=1}^{\infty} \mu_j a_j^2 \, dr \leq \int_0^1 \sum_{j=1}^{\infty} \mu_j^2 r^{n-3} \, dr ,
\end{equation}

\[ \mu_j^* = \max\{-\mu_j, 0\} , \]

and

\begin{equation}
(2.21) \quad \int_0^1 \sum_{j=1}^{\infty} \mu_j a_j^2 r^{n-3} dr = -\frac{2}{n-2} \int_0^1 r^{n-2} \sum_{j=1}^{\infty} \mu_j a_j a_j' \, dr \leq \frac{2}{n-2} \left[ \int_0^1 r^{n-1} \sum_{j=1}^{\infty} \mu_j a_j'^2 \, dr \right]^{\frac{1}{2}} \left[ \int_0^1 \sum_{j=1}^{\infty} \mu_j a_j^2 \, dr \right]^{\frac{1}{2}} ,
\end{equation}

so that

\[ \int_0^1 r^{n-3} \sum_{j=1}^{\infty} \mu_j^* a_j^2 \, dr \leq \frac{\mu_1^*}{(n-2)^2} \int_0^1 r^{n-1} \sum_{j=1}^{\infty} \mu_j^* (a_j')^2 \, dr ,
\]

using this in (2.20) we see that \( \int_{C_1} \xi L \zeta \leq 0 \ \forall \zeta \in C_0^2(C_1) \) provided

\begin{equation}
(2.22) \quad \frac{\mu_1^*}{(n-2)^2} \geq -1.
\end{equation}

Since \( \int_{C_1} \xi L \zeta = -\int_{C_1} (|\nabla \zeta|^2 - Q \zeta^2) \), we have thus shown

\begin{equation}
(2.23) \quad \int_{C_1} Q \zeta^2 \leq \int_{C_1} |\nabla \zeta|^2 \ \forall \zeta \in C_0^1(C_1)
\end{equation}

provided that (2.22) holds.
The condition (2.22) is equivalent to the requirement that all the roots $\gamma$ in (2.5)' are real. We then easily see (since a complex root $\gamma_1$ gives oscillatory solutions $\zeta_1 = \text{Re}(\gamma_1)$ for $L_\mathcal{C} \zeta_1 = 0$)

$$\inf_{\zeta \in c_0^2(c_1)} \int_{c_1} |\nabla \zeta|^2 - Q \zeta^2 < 0$$

if (2.22) fails. Thus we conclude that (2.23) holds if and only if (2.22) holds.

In case $\mathcal{C}$ is minimal of codimension 1 and $Q = r^{-2} |A|^2(\omega)$ ($A =$ second fundamental form of $\mathcal{C}$, restricted to $\mathcal{E}$) (2.23) is then exactly the stability inequality for $\mathcal{C}$ (cf. (1.1)). This extends routinely to the case of arbitrary codimension (as for Theorem 2). In this case the stability inequality requires

$$(2.24) \int_{c_1} A(\zeta) \cdot \xi \leq \int_{c_1} |\nabla \zeta|^2, \quad \forall \zeta \in c_0^1(c_1, \mathcal{E}^\perp),$$

and by the appropriate modifications of the argument above, we see that this is true if and only if (2.22) holds, where now $\mu_1$ is the minimum eigenvalue of the operator

$$L_\mathcal{E} \zeta \equiv \Delta_\mathcal{E} \zeta + A(\zeta),$$

where $\Delta_\mathcal{E}$ is the normal Laplacian for $\mathcal{E}$ (see [SJ]) and where $A$ is as in (2.18).

The condition (2.22) for stability (at least in the
codimensional 1 case) was first derived by J. Simons [SJ; Lemma 6.1.6]. R. Schoen has pointed out that it is equivalent to the requirement that $\Sigma$ be conformally equivalent to a positive scalar curvature manifold.

It is instructive to check the condition (2.22) for the codimension 1 examples $\Sigma = \left( \frac{1}{\sqrt{2}} S^P \right) \times \left( \frac{1}{\sqrt{2}} S^P \right)$. One readily checks in this case that $|A|^2(\omega) \equiv 2p$ on $\Sigma$, so that $\mu_1$ is the minimum eigenvalue of $\Delta_\Sigma + 2p$, which is trivially $-2p$. (The minimum eigenvalue of $\Delta_\Sigma$ is zero.) Since $n = 2p + 1$ in this case, the criterion (2.22) tells us that the cone $C$ over $\Sigma$ is stable if and only if

$$\frac{(-n+1)}{(n-2)^2} \geq -\frac{4}{n}$$

i.e.

$$n \geq 7.$$  

(Cf. [SJ].) Similar considerations show that if $\Sigma$ is a codimension 1 minimal submanifold of $S^n$, with second fundamental form having constant length $\kappa$, then the cone over $\Sigma$ is stable if and only if

$$\kappa \leq \frac{n-2}{2}.$$  

We note that the argument above actually establishes

$$\int_{C_1} \left( |\nabla \tau|^2 - \kappa \tau^2 \right) = \left( 1 - \frac{4\mu_1^-}{(n-2)^2} \right) \int_{C_1} (\partial \tau / \partial r)^2,$$
so that if we have strict inequality in (2.22), then

\[(2.25) \quad \int_{C_1} (|\nabla \zeta|^2 - Q_\zeta)^2 \geq \theta \int_{C_1} \left(\frac{\partial \zeta}{\partial r}\right)^2 \quad \forall \zeta \in C^1_0(C_1)\]

for some constant \( \theta \in (0, 1) \). Using the identity

\[\int_{E} \int_0^1 \frac{\partial}{\partial r} \left(\frac{r^{n-2} \zeta(r)}{r^2}\right) \, dr \, dw = 0, \]

so that

\[(2.26) \quad \int_{C_1} \frac{\zeta^2}{r^2} \leq \int_{C_1} \left(\frac{\partial \zeta}{\partial r}\right)^2 , \]

we then have

\[(2.26) \quad \int_{C_1} (|\nabla \zeta|^2 - Q_\zeta^2) \geq \theta' \int_{C_1} \frac{\zeta^2}{r^2} \quad \forall \zeta \in C^1_0(C_1) , \]

where \( \theta' > 0 \) is positive.

Notice that (by virtue of the standard representation of minimum eigenvalue in terms of Rayleigh quotient) (2.23) is equivalent to the assertion that the eigenvalue problem

\[
\begin{cases}
Lu + \frac{\lambda}{r^2} u = 0 & \text{on } C_{\varepsilon,1} \ni \{x \in \mathbb{C} : \varepsilon < |x| < 1\} \\
\frac{\partial u}{\partial r} = 0 & \text{on } \partial C_{\varepsilon,1}
\end{cases}
\]

(2.27)

has minimum eigenvalue \( \lambda_{\varepsilon} \geq 0 \) \( \forall \varepsilon > 0 \), while (2.26) is equivalent to \( \lambda_{\varepsilon} \geq \theta' > 0 \) (\( \theta' \) independent of \( \varepsilon \)).

That is, (2.22) \( \Rightarrow \lambda_{\varepsilon} \geq 0 \) \( \forall \varepsilon > 0 \) \( \Rightarrow \) (2.23); and strict inequality in (2.22) \( \Rightarrow \lambda_{\varepsilon} \geq \theta > 0 \) for some \( \theta \) independent of \( \varepsilon \)

\( \Rightarrow \) (2.26) for some \( \theta' > 0 \).

In view of this, and the fact that the \( u_{\Omega} \) obtained
in Theorems 1, 2 lie in $B_p$ ($B_p^1$ respectively) we now easily deduce the following theorem.

**THEOREM.** If $C$ is strictly stable in the sense that strict inequality holds in (2.22), then there is $\beta_0 > 0$ such that all the minimal graphs of Theorems 1, 2 are stable for $\beta \leq \beta_0$.

(If $\mu_1 = \frac{(n-2)^2}{4}$, then the minimal graphs of Theorems 1, 2 are stable in the weak sense that the stability inequality holds for $\zeta \in B_q$ (respectively $B_q^1$) with $\zeta = 0$ and $\Sigma$ and $\|\zeta\|_{B_q} \leq K$, provided $q > -\frac{(n-2)}{2}$ and provided $\beta$ is sufficiently small, depending on $K$).

§3. The question of classifying isolated singularities - uniqueness of tangent cones.

To facilitate the discussion of this section let us assume that $M$ has an isolated singularity in the following very strict sense:

**DEFINITION.** $M$ is said to have an isolated singularity at $0$ in the strong sense if $\text{sing } M \cap B_\rho = \{0\}$ for some $\rho > 0$ and if the second fundamental form $A$ of $M$ satisfies

\[(3.1) \quad |A(x)| \leq c/|x|, \quad 0 < |x| < \rho.\]

One would of course ultimately like to understand the general case when it is simply assumed $M \cap B_\rho = \{0\}$, but for our present purposes
the above assumption is convenient. Notice that (by the compactness and regularity theory developed in \[SS1\]) (3.1) will automatically hold for a seven dimensional stable embedded minimal hypersurface with an isolated singularity at 0.

In view of the examples of the previous section, one might be led to conjecture that M is always asymptotic to a minimal cone C near an isolated singular point at 0, at least in the sense

\[
(3.2) \lim_{\rho \to 0} \left( \sup_{x \in M \cap B_{\rho}} \frac{|x-y|}{\rho} \right) = 0
\]

for some minimal cone C. This would in fact be extremely illuminating, because it is elementary to check that (3.1), (3.2) imply that \( \text{sing } C = \{0\} \) and that the spherical nearest point projection \( \psi \) (taking a point \( y \in M \cap B_{\rho} \) to the point \( z \in C \cap B_{\rho} \) with least distance, measured in \( B_{\rho} \), from \( y \)) is a smooth covering projection. Thus we get a precise description of the exact nature of the isolated singularity at 0 in terms of the minimal cone C and an integer multiplicity (which is constant on the components of C).

Unfortunately, (3.2) appears very difficult to prove, even if we assume the strong condition (3.1). It is true that there always exist tangent cones in the sense that if \( \{\rho_k\} \) is a sequence of positive numbers converging to zero, then there is a subsequence \( \{\rho_{k'}\} \) and a cone C


[SL1] L. Simon, Survey lectures on minimal surfaces 


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