PARTIAL REGULARITY FOR SOLUTIONS OF VARIATIONAL PROBLEMS

N. Fusco and J.E. Hutchinson

We report here on some recent results of the authors [FH1,2] within the context of a general discussion of problems in the calculus of variations. Some results (those in [FH2]) were not included in the delivered lecture.

We will consider minima of functionals \( F \) of the form

\[
(1) \quad u \mapsto F[u] = \int_{\Omega} F(x,u,Du)
\]

where \( \Omega \subset \mathbb{R}^n \), \( \Omega \) is open, and \( u: \Omega \to \mathbb{R}^N \). It will always be assumed that \( F \) is a Caratheodory function, i.e. \( F = F(x,u,p) \) is measurable in \( x \) for all \( (u,p) \in \mathbb{R}^n \times \mathbb{R}^N \) and is continuous in \( (u,p) \) for almost all \( x \in \Omega \). This ensures that \( F(x,u,Du) \) is measurable if \( u \) is measurable.

Here will be interested in the general case \( n \geq 1 \) and \( N \geq 1 \).

If \( N = 1 \), one can obtain much stronger results, for this we refer to [Gl], [G2], [GT], [LU], and [M].

There are two questions of fundamental interest. First, one wants to show (subject to various boundary conditions) the existence of minima of \( F \) in suitable function classes. Second, one is interested in the regularity (i.e. smoothness) properties of such minimisers.

The existence problem in a general sense is solved as a standard consequence of the following result by Acerbi and Fusco [AF].

* Lecture delivered by the second author.
Theorem 1. Suppose \( F = F(x,u,p) \) is a Caratheodory function. Assume that
\[
0 \leq F(x,u,p) \leq \lambda (1 + |u|^m + |p|^m)
\]
for some \( m \geq 1 \).

Then the functional \( F \) is weakly sequentially lower semicontinuous in \( H^{1,m}(\Omega;\mathbb{R}^N) \) iff \( F \) is quasiconvex.

We say that \( F \) is quasiconvex if linear functions are local minimisers of the "frozen" functionals corresponding to \( F \). More precisely, \( F \) is quasiconvex if for a.e. \( x_0 \in \Omega \) and for all \( (u_0,p) \in \mathbb{R}^N \times \mathbb{R}^N \) one has
\[
\int_{\Omega} F(x_0,u_0,p) \leq \int_{\Omega} F(x_0,u_0,p + D\phi)
\]
for all \( \phi \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N) \).

For further discussion on the existence question we refer to the book [Gl] and the references therein.

We now discuss in somewhat more detail the regularity question for minima of \( F \).

Suppose \( F \) is a Caratheodory function, \( F = F(x,u,p) \) is \( C^2 \) in \( p \) for all \( (x,u) \in \Omega \times \mathbb{R}^N \), and \( F \) satisfies the following conditions

1. \( |p|^2 - 1 \leq F(x,u,p) \leq a(1 + |p|^2) \),
2. \( |F_{pp}(x,u,p)| \leq b \)
3. \( F_{pp} \xi^i \xi^j = F \sum_{\alpha \beta} \xi^i \frac{\partial^2 F}{\partial \alpha^\beta} \xi^j \geq \lambda |\xi|^2 \)
4. \( (1 + |p|^2)^{-1} F(x,u,p) \) is Hölder continuous in \( (x,u) \)

uniformly in \( p \). In other words
\[
|F(x,u,p) - F(y,v,p)| \leq c(1 + |p|^2)\omega(|x-y|^2 + |u-v|^2)
\]
where $w(t) \leq t^{\sigma}$, $0 < \sigma \leq \frac{1}{2}$, and $w$ is bounded, non-negative, concave, and increasing on $\{t \geq 0\}$.

Then we have the following result due to Giaquinta and Giusti [GG].

Theorem 2. Suppose $F$ is as in (1) and (2). Suppose $u \in W^{1,2}_{\text{loc}}(\Omega; \mathbb{R}^N)$ is a local minimum for $F$ (i.e. $F[u] \leq F[u + \phi]$ for all $\phi \in W^{1,2}_{\text{loc}}(\Omega; \mathbb{R}^N)$ with $\text{spt} \phi \subset \Omega$). Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{1,\alpha}_{\text{loc}}(\Omega_0)$ for some $0 < \alpha < 1$ and such that $H^N(\Omega_0 \setminus \Omega_0) = 0$. Moreover,

\begin{equation}
\Omega_0 = \{x_0 \in \Omega : \limsup_{r \to 0} |(Du)_{x_0, r}| < \infty \text{ and } \inf_{r \to 0} \left\{ \int_{B(x_0, r)} |Du - (Du)_{x_0, r}|^2 \right\} = 0 \}. \tag{3}
\end{equation}

The theorem is proved by ultimately establishing a local decay estimate in $\Omega_0$ of the form

\begin{equation}
\int_{B(x_0, r)} |Du - (Du)_{x_0, r}|^2 \leq cp^{2\alpha} \tag{4}
\end{equation}

as $p \to 0$, for some $\alpha > 0$. The key idea is to compare $u$ with the minimum $v$ in $B(x_0, r)$ of the frozen functional

$$w \mapsto \int_{B(x_0, r)} F(x_0, u, Dw)$$

with boundary condition

$$w \in u + W^{1,2}_{0}(B(x_0, r)).$$

In particular, one uses the fact that $w$, being a solution of a constant coefficient equation, satisfies a decay condition analogous to (4). Finally, one uses results of Companato [cf. [Gl, Chapter III]) to deduce the Hölder continuity of $u$ in $\Omega_0$ from (4).

It is an open question whether one can improve the dimension of
the singular set \( \Omega \sim \Omega_0 \). For particular classes of functionals this is indeed the case. On the other hand, one cannot generally expect everywhere regularity, as well-known counterexamples show. Again we refer to [Gl] for further discussion.

Aside from the question of the dimension of the singular set, there are some other gaps between the existence results which follow from Theorem 1 and the (partial) regularity results of Theorem 2.

In particular, the convexity condition of (2)(iii) implies quasic-convexity but not conversely; see [M, Chapter 4.4] and [Gl, Chapter IX.2]. However, it has recently been shown that if one replaces (2)(iii) by the requirement of strict quasic-convexity (see below), then one again has partial Hölder continuity of first derivatives of local minimisers.

One says that \( F \) is strictly quasiconvex if there exists \( \gamma > 0 \) such that for a.e. \( x_0 \in \Omega \), for all \( (u_0, p) \in \mathbb{R}^n \times \mathbb{R}^{nN} \), and for all \( \phi \in C^1_0(\mathbb{R}^n; \mathbb{R}^N) \), one has

\[
(2)(iii)^* \quad \int_{\Omega} [F(x_0, u_0, p) + \gamma |D\phi|^2] \leq \int_{\Omega} F(x_0, u_0, p + D\phi).
\]

The following theorem was proved by Evans [E] in case \( F \) depends only on \( p \), and then later for general \( F \) by Fusco and Hutchinson [FH] and also by Giaquinta and Modica [GM].

**Theorem 3** Under the same hypothesis as Theorem 2, but with (2)(iii) replaced by (2)(iii)*, we have that if \( u \in W^{1,2}(\Omega; \mathbb{R}^N) \) is a local minimum then \( u \in C^{1,\alpha}_{\text{loc}}(\Omega_0) \), for some \( 0 < \alpha < 1 \) and some open \( \Omega_0 \) satisfying \( H^N(\Omega \sim \Omega_0) = 0 \).

The proof in [E] was by means of a "blow-up" argument. The key new point was to establish the following Caccioppoli type estimate assuming (2)(iii)* rather than (2)(iii):
provided \( B(x_0, r) \subset \Omega, \ a \in \mathbb{R}^N, \ \xi \in \mathbb{R}^{nN}, \) and \(|\xi| \leq L\).

One is naturally tempted to extend the result in [E] (where \( F \) depends only on \( p \)) to general functionals \( F \) (depending on \( x \) and \( u \), as well as \( p \)) as follows. Suppose \( u \) is a local minimiser of \( F \). Try to obtain an estimate in \( \Omega_0 \) of the form

\[
(5) \quad \int_{B(x_0, \rho)} |Du - (Du)_{x_0, \rho}|^2 \leq c_0 2^\alpha
\]

by first estimating \( \int_{B(x_0, r)} |Du - Dv|^2 \), where \( v \) minimises

\[
\int_{B(x_0, r)} F(x, (u)_{x_0, r}, Dv) \quad \text{subject to} \quad v \in u + W^{1,2}_0(B(x_0, r)),
\]

and then by combining this with the estimate (5), with \( u \) replaced by \( v \), which estimate is proved in [E].

However, one cannot readily estimate \( \int_{B(x_0, r)} |Du - Dv|^2 \) with \( v \) as above, precisely because \( F_{\alpha, \beta}^{\gamma, \delta}(x, u, p) \) satisfies a Legendre-Hadamard condition \( F_{\alpha, \beta}^{\gamma, \delta} \geq \gamma |\xi|^2 |\eta|^2 \) rather than a Legendre condition \( F_{\alpha, \beta}^{\gamma, \delta} \geq \gamma |\xi|^2 \).

This problem is solved in [FH1] by invoking a lemma of Ekeland (cf. [Gl, Theorem 2.3, p.257]), from which one can deduce the existence for any \( B(x_0, r) \subset \Omega \) of a function \( v \) such that

\[
\int_{B(x_0, r)} |Dv - Du|^2 \leq r^{2\alpha}
\]

and \( v \) minimises the problem

\[
f \mapsto \int_{B(x_0, r)} F(x_0, (u)_{x_0, r}, Df) + c r^\beta \left( \int_{B(x_0, r)} |Df - Dv|^2 \right)^{\frac{1}{2}}
\]

\[
f \in v + W^{1,2}_0(B(x_0, r); \mathbb{R}^N)
\]
for some small positive $\alpha, \beta$. For further details see [FH1, §4].

The estimate (5) is obtained by means of a "blow-up" argument. Thus one supposes such an estimate is not true, blows up minimisers $v_m$ obtained as above in appropriate balls $B(x_m, r_m)$, and obtains a contradiction by passing to a limit of the $v_m$.

As remarked above, the results in [E], [GG] and [FH], allow a weakening of the hypotheses of Theorem 2 by replacing convexity by a strengthened form of quasiconvexity. Another natural weakening of the hypotheses of Theorem 2 is to replace the quadratic growth of $F(x,u,p)$ in the variable $p$, by a growth rate of order $|p|^m$ for some $m > 2$.

Motivated by a functional given by

$$F(x,u,p) = a(x,u)(1 + |p|^m), \quad m > 2,$$

we consider the following structural conditions to replace (2) (where $m > 2$):

(2) (i)' $|p|^m - 1 \leq F(x,u,p) \leq a(1 + |p|^m)$

(ii)' $|F_{pp}(x,u,p)| \leq b(1 + |p|^{m-2})$

(iii)' $F_{pp} \xi^2 \geq \lambda (1 + |p|^{m-2}) \xi^2$

(iv)' $(1 + |p|^m)^{-1} F(x,u,p)$ is Hölder continuous on

$(x,u)$ uniformly in $p$.

Then one can still prove $C^{1,\alpha}$ regularity (some $\alpha > 0$) on an open $\Omega_0$ with $H_n(\Omega \sim \Omega_0) = 0$, as in Theorem 2. Moreover, one can even replace (2)(iii)' by (2)(iii)*

(2)(iii)* $\int_{\Omega} [F(x_0,u_0,p_0) + \gamma(|D\phi|^2 + |D\phi|^m)]$

$$\leq \int_{\Omega} F(x_0,u_0,p_0 + D\phi)$$

for a.e. $x_0 \in \Omega$, for all $(u,p) \in R^N \times R^n$, for all $\phi \in C^1_0(R^n ; R^N)$, and for some $\gamma > 0$; see [E], [FH1] and [GM].
However, if one considers a functional given by (1) and

(6) \[ F(x,u,p) = a(x,u)|p|^m, \quad m \geq 2, \]

one sees that one should replace the structural condition (2)(iii)' by

(2)(iii)'' \[ F_{pp} \xi \xi \geq \lambda |p|^{m-2} \xi \xi. \]

Although (partial) regularity results are not known for such a

general class of functionals, there are some results. Uhlenbeck [U]

has shown complete \( C^{1,\alpha} \) (some small \( \alpha > 0 \))

regularity for

minimisers (even stationary points) of \( \int |Du|^m \) in case \( m \geq 2 \). This

has been extended to \( m > 1 \) by Tolksdorff [Tl]. Moreover, examples

show that one cannot expect \( C^{1,\alpha} \) regularity for all \( 0 < \alpha < 1 \)

(cf. [T2]).

In [FH2], partial regularity was shown for minimisers of

functionals corresponding to (6). More generally, we have the

following result.

**Theorem 4** Suppose \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is a local minimum for

(7) \[ F[u] = \int_{\Omega} [G^{\alpha\beta}(x,u)g_{ij}(x,u) Du^i D_{\beta} u^j]^{p/2} \]

where \( p \geq 2 \). Suppose \( G \) and \( g \) satisfy

\[ |\xi|^2 \leq G\xi \xi \leq M|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \]

\[ |\eta|^2 \leq G\eta \eta \leq M|\eta|^2 \quad \text{for all } \eta \in \mathbb{R}^n, \]

and \( G, g \) are \( C^{0,\sigma} \) on \( \Omega \times \mathbb{R}^n \).

Then \( u \in C^{1,\alpha}_{\text{loc}}(\Omega_0) \) for some \( 0 < \alpha < 1 \) and some \( \Omega_0 \subset \Omega \), where

\( H^{n-q}(\Omega_0) = 0 \) for some \( q > p \).

Moreover

(8) \[ \Omega_0 = \{ x_0 \in \Omega : \limsup_{r \to 0} |(u)_{x_0,r}| < \infty \} \]

and \( \liminf_{r \to 0} r^{p-n} \int_{|Du|^p = 0} \). \( \Box \)
The main idea in the proof is to first obtain an appropriate decay estimate for minima of functionals of the form

\[ u \mapsto \int \left[ G^{\alpha \beta} g_{i j} \partial_\alpha u \partial_\beta u^i \right]^{p/2}, \]

where \( [G^{\alpha \beta}] \), \( [g_{i j}] \) are constant inner products on \( \mathbb{R}^N \) and \( \mathbb{R}^n \) respectively.

By a change of coordinates, one reduces the problem to considering functionals of the form

\[ u \mapsto \int |Du|^p, \]

where \( |Du| = \left( \partial_\alpha u \partial_\beta u^i \right)^{1/2} \). Indeed, we work more generally with solutions of the Euler Lagrange equation

\[ \int |Du|^{p-2} Du D\phi = 0 \]

for all \( \phi \in W^{1,p}_0(\Omega; \mathbb{R}^N) \).

One might hope for an estimate on solutions \( u \) of (9) which has the form

\[ \int_{B(x_0, \tau R)} |Du - (Du)_{x_0, \tau R}|^p \leq c\tau^{p\alpha} \int_{B(x_0, R)} |Du - (Du)_{x_0, R}|^p \]

for all \( B(x_0, R) \subset \Omega \) and \( 0 < \tau < 1 \). However, it is not clear that such an estimate is true. What is done instead in [FH2] is to obtain an estimate of the form

\[ \Phi(x_0, \tau R) \leq c\tau^\alpha \psi(x_0, R) \quad \text{for } 0 < \tau < 1, \]

where one defines

\[ \Phi(x_0, \rho) = \int_{B(x_0, \rho)} |Du - (Du)_{x_0, \rho}|^p + |(Du)_{x_0, \rho}|^{p-2} \int_{B(x_0, \rho)} |Du - (Du)_{x_0, \rho}|^2, \]

whenever \( B(x_0, \rho) \subset \Omega \).

The proof of (10) uses earlier estimates of Uhlenbeck [U].

One finally proves Theorem 4 by comparing a minimum of (7) with a
minimum of the problem

$$
\left\{
\begin{array}{l}
\nu \mapsto \int_{B(x_0, R)} [G^{\alpha \beta}(x_0', (u)v(x_0', R) g_{ij}(x_0', (u)v(x_0', R) D^i \nu D^j \nu]^{p/2} \\
\nu \in u + W^{1,p}_0(B(x_0, R), \mathbb{R}^N),
\end{array}
\right.
$$

where $B(x_0, 2R) \subset \Omega$. One combines an estimate of the type (10) (with $u$ there replaced by $v$) together with an estimate of the form

$$
\int_{B(x_0, R)} |Du - Dv|^p \leq c^*_R \varepsilon
$$

for some small $\varepsilon > 0$, where here $c^*$ depends on $\int_{B(x_0, 2R)} |Du|^p$ and $\int_{B(x_0, R)} u$.

The resulting estimate which one obtains is

$$
\phi(x_0, TR) \leq c^{**}(TR)^\alpha
$$

for some small $\alpha > 0$ and all sufficiently small $T$, provided $B(x_0, 2R) \subset \Omega_0$, where $\Omega_0$ is defined in (8). Here $c^{**}$ has the same dependencies as $c^*$. By the usual Comparatio estimates it follows $u \in C^{1,\alpha}_{\text{loc}}(\Omega_0)$.

Finally, we remark that if in (7) the matrix $G$ does not depend on $u$, and $u$ is a locally bounded minimum, then the dimension of the singular set is at most $n - [q] - 1$ for some $q > p$ (q does not depend on u), where $[q]$ is the integer part of q. If $n \leq q + 1$, then $u$ can have at most isolated singularities. The proof is a modification of a similar argument in [GG].
REFERENCES


[FH1] N. Fusco J.E. Hutchinson, $c^{1,\alpha}$ partial regularity of functions minimizing quasiconvex integrals Manuscripta Mathematica 54 (1985), 121-143.


