WHEN ARE SINGULAR INTEGRAL OPERATORS BOUNDED?

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The aim of this talk is to survey some results concerning the \( L_2 \)-boundedness of singular integral operators, and in particular to present the \( T(b) \) theorem.

Let us consider one-dimensional singular integral operators \( T \) of the following type:

\[
(Tu)(x) = \text{p.v.} \int_{-\infty}^{\infty} K(x,y)u(y)dy
\]

where, for \( x, y \in \mathbb{R} \) with \( x \neq y \),

\[
\begin{align*}
|K(x,y)| &\leq c_0 |x-y|^{-1} \\
\frac{\partial K}{\partial x}(x,y) &\leq c_1 |x-y|^{-2} \\
\frac{\partial K}{\partial y}(x,y) &\leq c_2 |x-y|^{-2}.
\end{align*}
\]

Such \( T \) are called Calderón-Zygmund operators if \( \|T\|_2 \leq c \|\psi\|_2 \) for all \( \psi \in C^0_0(\mathbb{R}) \). We note first that an \( L_2 \)-estimate of this type is sufficient to prove a variety of bounds.

**THEOREM 1** (Calderón, Zygmund, Cotlar, Stein) Suppose \( T \) is a Calderón-Zygmund operator. If \( u \in L^p, 1 < p < \infty \), then \( Tu(x) \) is defined for almost all \( x \), and \( \|Tu\|_p \leq c \|u\|_p, 1 < p < \infty \). If
u \in L^\infty, then \|Tu\|_2 \leq c_0 \|u\|_\infty, where \|\cdot\|_* denotes the BMO norm and Tu is only defined modulo the constant functions.

In addition one has maximal-function estimates.

It has been a long-term program, initiated by Calderón, to determine whether certain classes of naturally occurring singular integral operators are Calderón-Zygmund operators. The best known case is when \( K(x,y) = k(x-y) \) with \( \hat{k} \in L^\infty(\mathbb{R}) \), where \( \hat{k} \) denotes the Fourier transform of \( k \). In this case, \( T = \hat{k}(D) \) where \( D = -i \frac{d}{dx} \) and \( \|Tu\|_2 \leq \|\hat{k}\|_\infty \|u\|_2 \). In particular, if \( K(x,y) = i\pi^{-1}(x-y)^{-1} \), then \( T = \text{sgn}(D) \), which is the Hilbert transform on \( \mathbb{R} \), appropriately scaled.

Another well-known class of kernels \( K_j \) give rise to the commutator integrals \( T_j \). These are defined by

\[
K_j(x,y) = \frac{1}{\pi} \frac{(g(x)-g(y))^j}{(x-y)^{j+1}}
\]

where \( g \) is a Lipschitz function. It was shown by Calderón that \( T_1 \) is bounded, and then by Coifman and Meyer that \( T_j \) is bounded for \( j > 1 \). Subsequently the bound

\[
\|T_j\|_2 \leq c(1+j)^4 \|g\|_\infty \|u\|_2
\]

was obtained by Coifman, McIntosh and Meyer [1].

It follows from these estimates for \( T_j \) that \( T_h \) is bounded, where \( T_h \) has kernel
\[ K_h(x, y) = \frac{i}{\pi}(h(x)-h(y))^{-1} , \]

with \( h \) a Lipschitz function such that \( \Re h'(x) \geq \lambda > 0 \) almost everywhere. For we can write \( h(x) = \rho(x-g(x)) \) with \( \rho > 0 \) and \( \|g''\|_\infty < 1 \), and then

\[ K_h(x, y) = \rho^{-1} \sum_{j=0}^{\infty} K_j(x, y) . \]

So

\[ \|T_h u\|_2 \leq \rho^{-1} \sum_{j=0}^{\infty} \|T_j u\|_2 \leq c_h \|u\|_2 . \]

The operator \( T_h \) arises as follows. The Cauchy integral on the Lipschitz curve \( \gamma \) parametrized by \( z = h(x) \) is

\[ C_\gamma U(z) = \frac{i}{\pi} \text{p.v.} \int_\gamma (z-\zeta)^{-1} U(\zeta) d\zeta . \]

On writing \( U(z(x)) = u(x) \), we get

\[ C_\gamma u(x) = \frac{i}{\pi} \text{p.v.} \int_{-\infty}^{\infty} K_h(x, y)u(y)h'(y)dy . \]

i.e.

\[ C_\gamma = T_h B \]

where \( B \) denotes multiplication by \( b = h' \). So \( C_\gamma \) is \( L_2 \)-bounded (though not itself a Calderón-Zygmund operator).
The original (unpublished) proof of the $L^2$-boundedness of $C_\gamma$ was quite different from that indicated above. It was shown that

$$\|D|^s C_\gamma u\|_2 \leq C_s \|D|^s u\|_2$$

when $0 < s < 1$, and hence that

$$\|D|^s T_h u\|_2 \leq C_s \|D|^s b^{-1} u\|_2.$$ 

Also, taking the dual of the above estimate with $b$ replaced by $\bar{b}$, we have

$$\|D|^{-s} T_h u\|_2 \leq C_s \|D|^{-s} u\|_2.$$ 

It was then shown that $T_h$ is $L^2$-bounded by interpolating these inequalities. This interpolation was achieved via a theorem of Kato which states that the domains of fractional powers of maximal accretive operators interpolate [4], and by proving a variant of the Kato square root problem, namely that

$$\frac{1}{2} \|D|^s b^{-1} |D|^s u\|_2 \leq c \|D|^s u\|_2.$$ 

Once the square root problem was solved, however, it was realized that the estimates used in its proof gave directly the boundedness of $T_j$ and hence of $T_h$ and $C_\gamma$.

Let us make some remarks about $C_\gamma$. Let $D_\gamma = \frac{1}{i} \frac{d}{dz} = b^{-1} D$.

Then $D_\gamma$ has spectrum in the double sector.
where \( \omega \) is large enough that \( S_\omega = \{ \zeta_1 - \zeta_2 | \zeta_1, \zeta_2 \in \gamma \} \). If the signum function is defined on \( S_\omega \) by

\[
\text{sgn } \zeta = \begin{cases} 
1 & \text{Re} \zeta > 0 \\
0 & \zeta = 0 \\
-1 & \text{Re} \zeta < 0
\end{cases}
\]

then \( C_\gamma = \text{sgn}(D_\gamma) \).

We remark that, for analytic functions \( \varphi \) on \( S_{\omega+\varepsilon}^{\circ} \) (the interior of \( S_{\omega+\varepsilon} \)) which decay suitably at \( \infty \), \( \varphi(D_\gamma) \) can be defined using resolvent integrals. On the other hand, if \( \varphi \) has inverse Fourier transform \( \hat{\varphi} \) which extends analytically to \( S_{\omega+\varepsilon}^{\circ} \) and decays suitably at \( \infty \), then

\[
\varphi(D_\gamma)U(z) = \int_\gamma \hat{\varphi}(z-\xi)U(\xi)d\xi.
\]

Let us go on. Subsequently to the operators \( T_j \) and \( T_h \) having been shown to be \( L_2 \)-bounded, David and Journé proved an intriguing theorem. We see from theorem 1 that if \( T \) is a Calderón-Zygmund operator then \( T(1) \in \text{BMO} \) and \( T^\#(1) \in \text{BMO} \). It is also clear that \( T \) satisfies the following weak boundedness property:

(2) there exists \( m \geq 0 \) and \( c \geq 0 \) such that

\[
|\langle Tu_1,u_2 \rangle| \leq cd
\]
for all $u_1, u_2 \in C_0^\infty(\mathbb{R})$ such that $u_1, u_2 \in C_0^\infty(\mathbb{R})$ where $u_1$ and $u_2$ have support in an interval of length $d$ and satisfy $|u_j^{(r)}| \leq d^{-r}$ for all $r \leq m$.

THEOREM 2. [2] Suppose $K$ satisfies (1). Then $T$ is a Calderón-Zygmund operator if and only if $T(1) \in \text{BMO}$, $T^*(1) \in \text{BMO}$ and $T$ satisfies (2).

As noted above, the "only if" part of this result is straightforward. But the "if" part is quite striking. We note that if $K(x,y) = -K(y,x)$ and (1) is satisfied, then (2) holds automatically. So in this case the $L_2$-boundedness is equivalent to $T(1) \in \text{BMO}$.

Theorem 2 can be used inductively to show that the commutator operators $T_j$ are bounded, but the bounds are not strong enough to imply that $T_h$ and $C_\gamma$ are bounded except when $h$ has a small Lipschitz constant.

Another interesting recent result is that of Lemarié. He proved a more general version of the following:

THEOREM 3. [5] Suppose that (1) is satisfied and that $T(b) = 0$ ($b \in \text{BMO}$) for some function $b \in L_\infty(\mathbb{R})$. Define $W$ by $W(u) = T(bu)$, and suppose that (2) holds with $T$ replaced by $W$. Then, for each $s \in (0,1)$, there exists $c_s$ such that

$$
\|D|^{S}W u\|_{2} \leq c_s \|D|^{S}u\|_{2}.
$$
As a corollary of this, Meyer and the author proved the following variant of David and Journé's theorem [6].

**THEOREM 4.** Suppose that \( b_1, b_2 \in L_\infty(\mathbb{R}) \) with \( \Re b_j(x) \geq \kappa > 0 \), that
\[
T(b_1) = 0 \quad \text{and} \quad T^*(\overline{b_2}) = 0,
\]
that (1) holds, and that (2) holds with \( T \) replaced by both \( TB_1 \) and \( B_2 T \) (where \( B_j \) is multiplication by \( b_j \)).

Then \( T \) is a Calderón-Zygmund operator.

This was proved by appealing to the square root problem in the same way as was originally done for the Cauchy integral.

Theorem 4 is a general theorem which includes the boundedness of the Cauchy integral as a special case, since \( T_h(h') = C_\gamma(1) = 0 \) (\( \in \) BMO) and \( C_\gamma \) satisfies (2). A more general result again, which includes both theorem 4 and theorem 2 as special cases, was subsequently proved by David, Journé and Semmes [3].

**THEOREM 5.** If the hypotheses of theorem 4 are weakened by replacing
\[
T(b_1) = 0 \quad \text{and} \quad T^*(\overline{b_2}) = 0 \quad \text{by} \quad T(b_1) \in \text{BMO} \quad \text{and} \quad T^*(\overline{b_2}) \in \text{BMO},
\]
then the conclusion remains valid.

Theorem 5 can be reduced to theorem 4 if, given \( \beta_1, \beta_2 \in \text{BMO} \), we can find Calderón-Zygmund operators \( L \) and \( M \) such that \( L(b_1) = \beta_1 \), \( L^*(\overline{b_2}) = 0 \), \( M(b_1) = 0 \) and \( M^*(\overline{b_2}) = \beta_2 \). To do this, let \( \gamma \) and \( \delta \) be the curves parametrized by \( z = h_1(x) \) and \( z = h_2(x) \), where \( h_1' = b_j \). Then define \( L \) by
\[
Lu = 2 \int_0^\infty \psi(tD_\delta)(\psi(tD_\delta)\beta_1)\psi(tD_\gamma)b_1^{-1}u\frac{dt}{t}.
\]
and define \( \mathbb{M}^* \) similarly. In this formula, \( \varphi \) and \( \psi \) denote the following functions:

\[
\varphi(\lambda) = (1+\lambda^2)^{-1} \quad \text{and} \quad \psi(\lambda) = \lambda(1+\lambda^2)^{-1}.
\]

So, if \( \zeta \in S_\omega \), where \( S_\omega \) was defined previously, then

\[
\varphi_t(\xi) = \begin{cases} 
\frac{1}{2\pi} e^{-\xi/t}, & \Re \xi > 0 \\
\frac{1}{2\pi} e^{\xi/t}, & \Re \xi < 0,
\end{cases}
\]

and

\[
\varphi(tD_\gamma)u(x) = \int_{\gamma} \varphi_t(z-\xi)U(\xi)d\xi,
\]
or

\[
\psi(tD_\gamma)u(x) = \int_{-\infty}^{\infty} \varphi_t(h_1(x)-h_1(y))b_1(y)u(y)dy.
\]

The operator \( \psi(tD_\delta) \) is defined similarly. Square function estimates for \( \psi(tD_\delta) \) can be obtained from the expansion for

\[
\psi(tD_\delta) = \psi(tB_2^{-1}D) = \psi(t\rho^{-1}(I-F)^{-1}D)
\]

in powers of \( F \) using the techniques of [1], where \( \rho \) is chosen so that \( \|F\| < 1 \). Proceeding in this way it can be shown that \( L \) is a Calderón-Zygmund operator. In doing this, we are generalizing the proof of the \( T(1) \) theorem given in [2] rather than following [3].

We conclude with the remark that theorems 1-5 remain valid in higher dimensions if the appropriate dependence on the dimension is...
included in (1) and (2). However many of the intervening comments are specifically one-dimensional.

REFERENCES


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