THE REGULARITY OF WEAK SOLUTIONS
TO PARABOLIC SYSTEMS

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There has been a lot of work devoted to the regularity of weak solutions to elliptic systems since De Giorgi [3] had shown in 1968 that his celebrated regularity result for elliptic equations cannot be extended to systems. Various methods such as the direct approach, the indirect approach and the hole-filling technique were developed to study the partial and everywhere regularity for quasilinear and nonlinear elliptic systems (see e.g. [4], [8] and the references cited there). It is reasonable to ask the following questions. How about the problem of regularity for parabolic systems? Are the results for elliptic systems still true for them? And do the methods mentioned above work in the parabolic case? Basically, the answers to these questions are positive.

Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( T > 0 \) and \( Q = \Omega \times (0,T) \). Denote \( z = (x,t) \), where \( x \in \mathbb{R}^n \), \( t \in \mathbb{R} \). For \( z_i = (x_i, t_i) \in \mathbb{R}^{n+1} \), \( i = 1,2 \), introduce the parabolic metric

\[
d(z_1, z_2) = \sqrt{1 + \left| \frac{|x_1 - x_2|^2 + |t_1 - t_2|^2}{2} \right|^2}.
\]

Throughout this paper we use the convention that repeated indices are to be summed for \( \alpha \) and \( \beta \) from 1 to \( n \), for \( i \) and \( j \) from 1 to \( N \), but not for \( k \).
In 1973, M. Giaquinta and E. Giusti [5] first studied the regularity of weak solutions for uniformly parabolic systems

\[ u_t^i - D_\alpha [A_{ij}^{\alpha\beta}(z,u)D_\beta u^j] = 0, \quad i = 1, \ldots, N, \quad z \in Q \]

with bounded and continuous coefficients \( A_{ij}^{\alpha\beta}(z,u) \), using the indirect approach. The direct approach, for which the \( L^p \)-estimate for the spatial derivatives of weak solutions is crucial, has been carried on in S. Campanato [1,2], M. Giaquinta and M. Struwe [7], under strictly controlled and quadratic growth conditions respectively. This work generalized the partial regularity result of [5]. By means of the hole-filling technique, an everywhere regularity result for diagonal parabolic systems with quadratic growth was obtained by M. Struwe [9] and improved by M. Giaquinta and M. Struwe [6].

S. Campanato considered first (in [1]) the quasilinear system

\[ (1) \quad u_t^i - D_\alpha [A_{ij}^{\alpha\beta}(z,u)D_\beta u^j + a_i^\alpha(z,u)] = B_i(z,u,Du), \quad i = 1, \ldots, N \]

and then (in [2]) the nonlinear system

\[ (2) \quad u_t^i - D_\alpha A_{ij}^{\alpha}(z,u,Du) = B_i(z,u,Du), \quad i = 1, \ldots, N. \]

He assumed, among other conditions, that \( A_{ij}^{\alpha} \) and \( B_i \) have the following growth, which he called strictly controlled:
\[ |A_1^{\alpha}(z,u,p)| \leq C(|p| + |u|^{\frac{2}{\gamma}} + f_{1\alpha}) \]

\[ |B_1(z,u,p)| \leq C(|p| + |u|^{\gamma-1} + f_1) \]

with \( f_{1\alpha} \) and \( f_1 \) belonging to some \( L^p \) spaces and

\[ 1 \leq \gamma < \begin{cases} 
\frac{2(n+2)}{n} & \text{if } n \geq 2, \\
4 & \text{if } n = 1.
\end{cases} \]

He established the \( L^p \)-estimate for the spatial derivatives of weak solutions of system (1) and (2), similar to those in elliptic case, and showed partial Hölder continuity of every solution of system (1) or (2), with a singular set \( Q_0 \), closed in \( Q \), such that

\[ H^n(Q_0) = 0 \text{ or } \text{meas } Q_0 = 0, \]

where \( H^n \) stands for the Hausdorff measure (relative to the parabolic metric \( d \)) of dimension \( n \), and \( \text{meas} \) the Lebesgue measure of dimension \( n+1 \).

For systems (1) with quadratic growth, M. Giaquinta and M. Struwe [7] proved the \( L^p \)-estimate and partial Hölder continuity, and provided the singular set \( Q_0 \) with a refined estimate:

\[ H^{n-\varepsilon}(Q_0) = 0 \text{ for some } \varepsilon > 0. \]

A few questions about the results mentioned above can be raised.
(i) Can we get $L^p$-estimate for system (2) with controllable growth for the critical case $\gamma = 2(n+2)/n$ and without exception of $n=1$? Does there also exist an $L^p$-estimate for nonlinear systems (2) (not only quasilinear systems (1)!) under the natural growth conditions?

(ii) Is every weak solution of system (1) or (2) partially Hölder-continuous under both natural and controllable growth conditions including the case $\gamma = 2(n+2)/n$? How about the Hausdorff measure of its singular set?

(iii) For what sort of parabolic systems does any weak solution have everywhere regularity?

(iv) Under what conditions can further regularity be obtained?

Here I wish to briefly describe some results by myself [10], which have partly answered the above questions.

Consider parabolic systems

\begin{equation}
\frac{u_t^i}{\alpha} - D A_i^\alpha(z,u,Du) = B_i(z,u,Du) , \quad i = 1, \ldots, N , \quad z \in \Omega .
\end{equation}

Assume that the controllable growth conditions hold, i.e.

\begin{equation}
A_i^\alpha(z,u,p) p_i^\alpha \geq \lambda |p|^{2(n+2)} - C |u|^{n} - f^{2} , \quad \lambda > 0 , \quad 0 \leq f \in L^\sigma(\Omega) ,
\end{equation}

\begin{equation}
|A_i^\alpha(z,u,p)| \leq C(|p|^{n} + |u|^{n} + f_{i\alpha}) , \quad 0 \leq f_{i\alpha} \in L^\sigma(\Omega) ,
\end{equation}
For \( z_0 = (x_0, t_0) \in Q \), and \( R > 0 \), we denote

\[
B_R = B(x_0, R) = \{ x \in \mathbb{R}^n : |x - x_0| < R \},
\]

\[
I_R = I(t_0, R) = \{ t : t_0 - R^2 < t < t_0 \},
\]

\[
Q_R = Q(z_0, R) = B(x_0, R) \times I(t_0, R)
\]

and

\[
\frac{1}{\text{meas}_{Q_R}} \int_{Q_R} \sim = \int_{Q_R} \sim.
\]

**THEOREM 1.** Suppose that (4), (5) and (6) hold with \( \sigma > 2 \), \( r > \frac{2(n+2)}{n+4} \). If \( u \in L^2(0, T; H^1(\Omega, \mathbb{R}^n)) \cap L^\infty(0, T; L^2(\Omega, \mathbb{R}^N)) \) is a weak solution of system (3), then there exists an exponent \( p > 2 \) such that \( \|Du\| \in L^p_{\text{loc}}(Q) \). Moreover, for \( Q(z_0, R) \subset Q(z_0, 4R) \subset Q \), we have

\[
\left[ \int_{Q_R} \left( \|Du\|^2 + |u|^{n+2} \right)^{p/2} \right]^{1/p} \leq C \left[ \int_{Q_R} \left( \|Du\|^2 + |u|^{n+2} \right) \right]^{1/2}
\]

\[
+ \left[ \int_{Q} \left( |u|^p \right) \right]^{1/p},
\]

provided \( R < R_0 \), where \( C \) and \( R_0 \) are constants depending on \( u \), and
Next we consider system (3) under natural growth conditions:

\[ F = f + \sum_{i,\alpha} f_{i\alpha} + \sum_{i}^{\frac{n+2}{n+4}} f_{i} \]

(7) \( A_{1}^{\alpha}(z,u,p) p_{\alpha}^{1} \geq \lambda |p|^{2} - t^{2}, \quad 0 \leq f \in L^{2}(Q) \).

(8) \( |A_{1}^{\alpha}(z,u,p)| \leq C(|p| + f_{i\alpha}), \quad 0 \leq f_{i\alpha} \in L^{2}(Q) \).

(9) \( |B_{1}(z,u,p)| \leq C(|p|^{2-\delta} + f_{1}), \quad 0 < \delta < \frac{n}{n+2}, \quad 0 \leq f_{1} \in L^{r}(Q) \)

or

(9') \( |B(z,u,p)| \leq a|p|^{2} + b, \quad 0 \leq b \in L^{r}(Q) \) and \( 2aM < \lambda \), where

\[ M = \sup_{Q} |u| \]

**Theorem 2.** Suppose that (7), (8) and (9) or (9)' hold with \( \sigma > 2 \) and \( r > 1 \), and that \( u \in L^{2}(0,T; H^{1}(\Omega,\mathbb{R}^{N})) \cap \cap L^{\infty}(Q,\mathbb{R}^{N}) \) is a weak solution of system (3), with \( \sup_{Q} |u| = M \). Then there exists \( p > 2 \) such that

\( |Du| \in L^{p}_{loc}(Q) \). Moreover, for \( Q(z_0,R) \subset Q(z_0,4R) \subset Q \), we have

\[
\left[ \frac{1}{Q_{R}} \right]^{1/p} \leq C \left[ \frac{1}{Q_{4R}} \right]^{1/2} + \left[ \frac{1}{Q_{4R}} \right]^{1/p},
\]

provided \( R < R_{0} \), where \( C \) and \( R_{0} \) are constants depending on \( u \), and
Applying the \( L^p \)-estimates to quasilinear systems

\[
F = f + \sum_{i, \alpha} f_{i \alpha} + \sum_{i} f_{i} \quad (or \quad \beta^{1/2} ).
\]

(10) \( u_t^i - D_{\alpha} \left[ \alpha_{ij}^\alpha(z,u) \partial_{\beta} u_j^i + \alpha_i^\alpha(z,u) \right] = \beta_i(z,u,Du), \; i = 1, \ldots, N, \; z \in Q \)

with assumptions:

(11) \( \lambda |\xi|^2 \leq A_{ij}^\alpha(z,u) \xi_i^\alpha \xi_j^\beta \leq \lambda |\xi|^2, \; \lambda \geq \lambda > 0 \),

(12) There exists a continuous, bounded, nondecreasing and concave function \( \omega \) such that \( \omega(0) = 0 \) and

\[
\sum_{i} |A_{ij}^\alpha(z,u) - A_{ij}^\alpha(z_0,u_0)|^2 \leq \omega(|z-z_0|^2 + |u-u_0|^2)
\]

for any \( z, z_0 \in Q \) and any \( u, u_0 \in \mathbb{R}^N \),

\[
(13) \; |f_{i \alpha}^\alpha(z,u)| \leq C(|u|^n + |f_{i \alpha}|), \; f_{i \alpha} \in L^\sigma(Q),
\]

\[
(14) \; |B_{i}(z,u,p)| \leq C(|p|^{n+2} + |u|^{n+2} + |f_{i}|), \; f_{i} \in L^\tau(Q),
\]

we can get the partial Hölder continuity for their solutions.

Precisely, we have

**Theorem 3.** Suppose that \( u \in L^2(0,T;H^1(\Omega,\mathbb{R}^n)) \cap L^\infty(0,T;L^2(\Omega,\mathbb{R}^n)) \) is a weak solution of system (10) and that (11)-(14) hold with \( \sigma > n+2 \) and \( \tau > \frac{(n+2)^2}{n+4} \). Then \( u \) is Hölder continuous in an open set \( Q' \subset Q \).
and \( H^{n+2-p}(Q \setminus Q') = 0 \), where \( p > 2 \) is the number appearing in Theorem 1.

As mentioned previously, M. Giaquinta and M. Struwe [7] have already proved the partial Hölder continuity for system (10) with quadratic growth. Theorem 3 is not an immediate consequence of that result, although controllable growth is much weaker than quadratic growth. This is because in the former case, weak solutions are not required to be bounded.

As far as everywhere regularity is concerned, we have the following Theorems 4 and 5.

Consider the parabolic system of triangular form, i.e., system

\[
\begin{align*}
(15) & \quad u_t^i - D_\alpha [A_{ij}(z,u)D_{\beta} u^j + a_1^i(z,u)] = B_1(z,u,Du), \quad i = 1, \ldots, N, \quad z \in Q \\
\intertext{with} & A_{ij}^\alpha \text{ satisfying} \\
(16) & \quad A_{ij}^\alpha(z,u) = 0 \quad \text{when} \quad j > i, \quad \lambda |\xi|^2 \leq A_{kk}^\alpha(z,u)t^\alpha \xi \eta \leq \Lambda |\xi|^2 \\
\intertext{for all} & i, j, k = 1, \ldots, N, \quad \text{all} \quad z \in Q, \quad u \in \mathbb{R}^N, \quad \xi \in \mathbb{R}^n \quad \text{and some} \quad \Lambda \geq \lambda > 0.
\end{align*}
\]

**THEOREM 4.** Under controllable conditions (13) and (14) with \( \sigma > n+2, \quad \tau > \frac{n}{2} + 1 \), every solution \( u \in L^2(0,T; H^1(\Omega; \mathbb{R}^N)) \cap L^\infty(0,T; L^2(\Omega; \mathbb{R}^N)) \) of triangular parabolic system (15), (16) is (locally) Hölder continuous in \( Q \).
THEOREM 5. Under the natural growth conditions

\[ |a_{i\alpha}^\alpha(z,u)| \leq \delta_{i\alpha} \in L^{\sigma}(Q) , \sigma > n+2 , \]

\[ |B_i(z,u,p)| \leq C|p|^{2-\delta} + f_i , \]

\[ 0 < \delta < \frac{n}{n+2} , f_i \in L^\tau(Q) , \tau > \frac{n}{2} + 1 , \]

every solution \( u \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(Q,\mathbb{R}^N) \) of a triangular parabolic system (15), (16) is (locally) Hölder continuous in \( Q \).

Unfortunately, we have found no way out yet to show the everywhere regularity for the triangular parabolic system (15), (16) with quadratic growth (9)' As for the system of diagonal form

\[ \frac{\partial u_i}{\partial t} - \mathcal{D}_{\alpha}[a^{\alpha\beta}(z)\mathcal{D}_{\beta}u_i] = B_i(z,u,Du) , i = 1, \ldots, N , z \in Q . \]

M. Giaquinta and M. Struwe [6] proved the following result:

Let \( u \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(Q,\mathbb{R}^N) \) with \( \sup \|u\| = M \) be a weak solution of system (17) which satisfies the conditions:

\[ \lambda |\xi|^2 \leq a^{\alpha\beta} \xi_{\alpha} \xi_{\beta} \leq \Lambda |\xi|^2 , \Lambda \geq \lambda > 0 , \]

\[ |B(z,u,p)| \leq \Lambda |p|^2 + b , \]

\[ aM < \lambda , b \in L^\tau(Q) , \tau > \frac{n}{2} + 1 . \]

Then \( u \) is (locally) Hölder continuous in \( Q \).
Let us come back to the general quasilinear system (1) and state a further regularity result.

THEOREM 6. Suppose $u \in C^{\mu, \mu/2}_{\text{loc}}(Q, \mathbb{R}^N)$, $0 < \mu \leq 1$, is a weak solution of system (1). Assume that

$$A_{ij}^{\alpha \beta}(z,u) t^i t^j_{\alpha \beta} \geq \lambda |t|^2$$

for some $\lambda > 0$,

$$A_{ij}^{\alpha}(z,u) \text{ and } a_{i}^{\alpha}(z,u) \text{ are H"{o}lder continuous with exponent } \nu \text{ in } x,u \text{ and } \nu/2 \text{ in } t,$$

and

$$|B(z,u,p)| \leq C(1+|p|^2).$$

Then the derivatives $Du$ of $u$ are (locally) H"{o}lder continuous in $Q$ with the same exponents $\nu$ and $\nu/2$ in $x$ and $t$ respectively.

Starting with Theorem 6, higher regularity can be obtained from the linear theory.

In conclusion, I wish to point out that M. Struwe [9] has given a global regularity result. But, in general, the regularity up to the boundary has not been studied extensively.
REFERENCES


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