### 1. ADJOINT CONSIDERATIONS

A useful way of studying a complex Banach space $X$ and a bounded linear operator $T$ on $X$ is to consider the **adjoint space**

$$X^* = \{ x^* : X \to \mathbb{C}, x^* \text{ is conjugate linear and continuous} \}$$

of $X$ and the **adjoint operator** $T^*$ associated with $T$. In this section we develop these concepts. This is done in such a way as to make the well-known Hilbert space situation a particular case of our development.

For $x^* \in X^*$ and $x \in X$, we denote the value of $x^*$ at $x$ by

$$\langle x^*, x \rangle .$$

Then we easily see that for $x^*$ and $y^*$ in $X^*$, $x$ and $y$ in $X$ and $t \in \mathbb{C}$,

\[
\begin{align*}
\langle x^*, x+y \rangle &= \langle x^*, x \rangle + \langle x^*, y \rangle , \\
\langle x^*, tx \rangle &= t \langle x^*, x \rangle , \\
\langle x^* + y^*, x \rangle &= \langle x^*, x \rangle + \langle y^*, x \rangle , \\
\langle tx^*, x \rangle &= t \langle x^*, x \rangle .
\end{align*}
\]

We say that $\langle , \rangle$ is the **scalar product** on $X^* \times X$. For the sake of convenience, we introduce the following notation:

\[
\langle x, x^* \rangle = \langle x^*, x \rangle , \text{ } x \text{ in } X \text{ and } x^* \text{ in } X^* .
\]

For $x^*$ in $X^*$, let

$$\| x^* \| = \sup \{ | \langle x^*, x \rangle | : x \in X, \| x \| \leq 1 \} .$$
This defines a norm on $X^*$ and makes it a Banach space. We have the fundamental inequality:

(1.3) $|<x^*, x>| = |<x, x^*>| \leq \|x^*\| \|x\|$, $x^*$ in $X^*$ and $x$ in $X$.

Many books on functional analysis consider the dual space $X' = \{x' : X \to \mathbb{C} : x'$ is linear and continuous\}$ of $X$ instead of the adjoint space $X^*$. We prefer the framework of the adjoint space because in case $X$ is a Hilbert space, $X^*$ can be linearly identified with $X$ itself, as we shall see later. In any event, we remark that $x' \in X'$ iff its complex conjugate $\overline{x'} \in X^*$. This allows us to transfer many well-known results about $X'$ to $X^*$, such as the following basic extension result.

**Proposition 1.1** (Hahn-Banach theorem) Let $Y$ be a subspace of $X$ and $y^* \in Y^*$. Then there is $x^* \in X^*$ such that $x^*|_Y = y^*$ and $\|x^*\| = \|y^*\|$.

**Proof** Since $\overline{y^*} \in Y'$, there is $x' \in X'$ with $x'|_Y = \overline{y^*}$ and $\|x'\| = \|\overline{y^*}\| = \|y^*\|$, by the Hahn-Banach extension theorem ([L], 7.6). The proof is complete if we let $x^* = x^\top$. //

**Corollary 1.2** If $0 \neq a \in X$, then there is $x^* \in X^*$ with $<x^*, a> = \|a\|$ and $\|x^*\| = 1$. More generally, if $Y$ is a closed subspace of $X$ and $a \in Y$, then there is $x^* \in X^*$ such that $<x^*, a> = \text{dist}(a, Y)$, $\|x^*\| = 1$ and $x^*|_Y \equiv 0$. 
Proof The first result follows by letting $Y = \text{span}(a)$ and

$$\langle y^*, ta \rangle = t\|a\|$$

in Proposition 1.1. The second part can be proved by considering the quotient space $X / Y$ with the quotient norm

$$\|x + y\| = \inf \{ \|x + y\| : y \in Y \} = \text{dist}(x, Y)$$

for $x \in X$, and then using the first part. //

The above result is useful in expressing the duality between $X$ and $X^*$: Just as $\langle x^*, x \rangle = 0$ for all $x \in X$ implies, by definition, that $x^* = 0$, we see that $\langle x, x^* \rangle = 0$ for all $x^* \in X^*$ implies, by the above corollary, that $x = 0$. Moreover, just as we have by definition, for $x^* \in X^*$,

$$\|x^*\| = \sup \{ |\langle x^*, x \rangle| : x \in X, \|x\| \leq 1 \},$$

we see by (1.3) and the above corollary that for $x \in X$,

$$\|x\| = \sup \{ |\langle x, x^* \rangle| : x^* \in X^*, \|x^*\| \leq 1 \}.$$

For a subset $E$ of $X$, we define the annihilator $E^\perp$ of $E$ to be the following subset of $X^*$:

$$E^\perp = \{ x^* \in X^* : \langle x^*, x \rangle = 0 \text{ for all } x \in E \}.$$

It is easy to see that $E^\perp$ is, in fact, a closed subspace of $X^*$. The concept of an annihilator will be used later in relating the range of a bounded linear map to the zero space of its adjoint.

Let $X$ and $Y$ be complex Banach spaces, and let $\text{BL}(X, Y)$ denote the set of all bounded linear maps from $X$ to $Y$. For $T \in \text{BL}(X)$, the operator norm of $T$ is defined as follows:

$$\|T\| = \sup \{ \|Tx\| : x \in X, \|x\| \leq 1 \}.$$
Two important subspaces related to $T$ are the null space of $T$:

$$Z(T) = \{ x \in X : Tx = 0 \},$$

and the range of $T$:

$$R(T) = \{ y \in Y : y = Tx \text{ for some } x \in X \}.$$

For $T \in \text{BL}(X,Y)$ and $y^* \in Y^*$, we see that $y^* T \in X^*$. We denote this element of $X^*$ by $T y^*$. Thus, the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{T^*} & & \downarrow{y^*} \\
Y^* & \xleftarrow{y} & \text{C}
\end{array}$$

The adjoint $T^*$ of $T$ is the map from $Y^*$ to $X^*$ defined by

$$\langle T y^*, x \rangle = \langle y^*, Tx \rangle \text{ for } y^* \in Y^*, x \in X.$$

Taking conjugates, and using the notation (1.2), we have

$$\langle Tx, y^* \rangle = \langle x, T^* y^* \rangle \text{ for } x \in X, y^* \in Y^*.$$

**Proposition 1.3**  
(a) For $T \in \text{BL}(X,Y)$, we have $T^* \in \text{BL}(Y^*,X^*)$ and $\|T^*\| = \|T\|.

(b) For $T, S \in \text{BL}(X,Y)$ and $t \in \mathbb{C}$, we have

$$(T + S)^* = T^* + S^* \text{ and } (tT)^* = \overline{t} T^*.$$

Thus, $T \mapsto T^*$ is a conjugate linear isometry of $\text{BL}(X,Y)$ into $\text{BL}(Y^*,X^*)$.

(c) The null space of $T^*$ equals the annihilator of the range of $T$:

$$Z(T^*) = R(T)^\perp.$$
(d) Let \( Z \) be a complex Banach space, and \( U \in \text{BL}(Y,Z) \). Then
\[
(UT)^* = T^*U^*.
\]

**Proof** (a) \( T^* \) is clearly linear. Also,
\[
\|T^*\| = \sup\{\|T^* y^*\| : y^* \in Y^*, \|y^*\| \leq 1\}
= \sup\{|\langle y^*, Tx \rangle| : y^* \in Y^*, \|y^*\| \leq 1, x \in X, \|x\| \leq 1\}
= \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}
= \|T\|.
\]

(b) The proof of this part is easy. For example, one quickly shows that for every \( y^* \in Y^* \), and \( x \in X \),
\[
\langle (T+S)^* y^*, x \rangle = \langle (T^*+S^*) y^*, x \rangle.
\]

(c) We have \( y^* \in Z(T^*) \) if and only if \( \langle y^*, Tx \rangle = \langle T^* y^*, x \rangle = 0 \) for every \( x \in X \) if and only if \( y^* \in \text{R}(T)^\perp \).

(d) For \( z^* \in Z^* \) and \( x \in X \), we have
\[
\langle (UT)^* z^*, x \rangle = \langle z^*, UTx \rangle = \langle U^* z^*, Tx \rangle = \langle T^* U^* z^*, x \rangle.
\]

Hence the result. //

Special Case of a Hilbert Space.

Let \( X \) be a Hilbert space with the inner product \( \langle \cdot, \cdot \rangle_X \), and let \( \|x\| = (\langle x, x \rangle_X)^{1/2} \) for \( x \in X \). Given \( x^* \in X^* \), define \( f : X \rightarrow \mathbb{C} \) by
\[
f(x) = \langle x, x^* \rangle, \quad x \in X.
\]

Then \( f \) is a continuous linear functional on \( X \) of norm \( \|x^*\| \). The Riesz representation theorem ([L], 24.2) shows that there is unique \( y \in X \) such that
\[ \langle x, x^* \rangle = f(x) = \langle x, y \rangle_X \]

for all \( x \in X \); moreover, \( \|y\| = \|f\| = \|x^*\| \). The correspondence \( x^* \mapsto y \) of \( X^* \) with \( X \) is, thus, a linear isometry onto. Whenever \( X \) is a Hilbert space, we shall, from now on, identify \( X^* \) with \( X \) via the above correspondence, and drop the suffix \( X \) in the inner product notation \( \langle \cdot, \cdot \rangle_X \) without any ambiguity.

Let \( A : X \to X \) be a linear map. The generalized polarization identity

\[
4\langle Ax, y \rangle = \langle A(x+y), x+y \rangle - \langle A(x-y), x-y \rangle + i\langle A(x+iy), x+iy \rangle - i\langle A(x-iy), x-iy \rangle ,
\]

where \( x \) and \( y \) belong to \( X \), is often useful.

For a subset \( E \) of the Hilbert space \( X \), the annihilator \( E^\perp = \{ y \in X : \langle x, y \rangle = 0 \text{ for all } x \in E \} \) consists of all elements of \( X \) which are orthogonal to \( E \). The double annihilator \( E^{\perp \perp} \) has a nice characterization: If \( F \) denotes the closure of the linear span of \( E \), then

\[
E^{\perp \perp} = F .
\]

It is easy to check that \( F \) is contained in \( E^{\perp \perp} \). On the other hand, suppose for a moment that there is some \( a \) in \( E^{\perp \perp} \), but not in \( F \). Then by Corollary 1.2, there is \( x^* \in X^* \) such that \( x^* \big|_F = 0 \) but \( \langle x^*, a \rangle = 1 \), i.e., there is \( y \in X \) such that \( \langle z, y \rangle = 0 \) for all \( z \in F \), but \( \langle a, y \rangle = 1 \). This is impossible since \( y \in E^\perp \) and \( a \in E^{\perp \perp} \) so that \( \langle a, y \rangle = 0 \).
For $T \in BL(X)$, the adjoint operator $T^* \in BL(X)$ is characterized by

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \text{ for all } x \text{ and } y \text{ in } X.$$ 

In addition to $Z(T^*) = R(T)^{\perp}$, as noted in Proposition 1.3(c), we also have

$$Z(T) = R(T^*)^{\perp},$$

when $X$ is a Hilbert space. This follows since $x \in Z(T)$ if and only if $0 = \langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $y \in X$ if and only if $x \in R(T^*)^{\perp}$. Thus, $T$ (resp., $T^*$) is one to one if and only if the range of $T^*$ (resp., $T$) is dense in $X$.

The norms of the operators $T$ and $T^*$ are related by the $B^*$-algebra condition

$$\|T^* T\| \leq \|T\| \|T^*\| \leq \|T\|^2.$$ 

This can be proved as follows.

$$\|T^* T\| \leq \|T^*\| \|T\| \leq \|T\|^2 = \sup\{\|Tx\|^2 : x \in X, \|x\| \leq 1\} = \sup\{\langle Tx, Tx \rangle : x \in X, \|x\| \leq 1\} \leq \|T^* T\|.$$ 

If $T^*$ commutes with $T$, i.e., $T^* T = TT^*$, we say that $T$ is normal; and if $T^* = T$, we say that $T$ is self-adjoint. It is clear that every self-adjoint operator is normal.
For $x \in X$ and $T \in \text{BL}(X)$, we have

$$
\|Tx\|^2 - \|T^*x\|^2 = \langle Tx, Tx \rangle - \langle T^*x, T^*x \rangle
= \langle (T^*T - TT^*)x, x \rangle.
$$

Hence it follows by using the generalized polarization identity that

\begin{equation}
(1.8) \quad T \in \text{BL}(X) \text{ is normal if and only if } \|Tx\| = \|T^*x\|
\text{ for all } x \in X.
\end{equation}

For a self-adjoint operator $T$, we have

$$
\langle Tx, x \rangle = \langle x, T^*x \rangle
= \langle x, Tx \rangle
= \langle Tx, x \rangle
$$

for all $x \in X$, so that $\langle Tx, x \rangle$ is real. Conversely, let $\langle Tx, x \rangle$ be real for all $x \in X$. Then for $x, y \in X$, the generalized polarization identity shows that,

$$
4\langle Tx, y \rangle = \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle
+ i\langle T(x+iy), x+iy \rangle - i\langle T(x-iy), x-iy \rangle
= \langle x+y, T(x+y) \rangle - \langle x-y, T(x-y) \rangle
+ i\langle x+iy, T(x+iy) \rangle - i\langle x-iy, T(x-iy) \rangle
\quad (\text{since } \langle Tz, z \rangle \text{ is real for all } z \in X)
= \langle T^*(x+y), x+y \rangle - \langle T^*(x-y), x-y \rangle
+ i\langle T^*(x+iy), x+iy \rangle - i\langle T^*(x-iy), x-iy \rangle
= 4\langle T^*x, y \rangle.
$$

Hence $T^* = T$, i.e., $T$ is self-adjoint. Thus,

\begin{equation}
(1.9) \quad T \in \text{BL}(X) \text{ is self-adjoint if and only if } \langle Tx, x \rangle
\text{ is real for all } x \in X.
\end{equation}
Examples of adjoint spaces and operators

(i) Let $X$ be an $n$ dimensional space with $1 \leq n < \infty$, and let $x_1, \ldots, x_n$ be an ordered basis for $X$. Then for $x$ in $X$, we have

\[ x = \sum_{j=1}^{n} t_j(x) x_j, \]

where $t_j(x) \in \mathbb{C}$, $j = 1, \ldots, n$, is uniquely determined by $x$. If we let

\[ \langle x_j^*, x \rangle = t_j(x), \quad j = 1, \ldots, n, \]

then $x_1^*, \ldots, x_n^*$ is an ordered basis for $X^*$ and we have

\[ \langle x_j^*, x_i \rangle = \delta_{i,j}, \quad i, j = 1, \ldots, n, \]

where $\delta_{i,j}$ is the Kronecker symbol: $\delta_{i,j}$ equals 0 if $i \neq j$, and equals 1 if $i = j$. This basis is called the basis of $X^*$ which is adjoint to the given basis $x_1, \ldots, x_n$ of $X$.

For $x$ in $X$ and $x^*$ in $X^*$, we have

\[ x = \sum_{j=1}^{n} \langle x, x_j^* \rangle x_j + \cdots + \langle x, x_n^* \rangle x_n, \]

(1.9)

\[ x^* = \sum_{j=1}^{n} \langle x^*, x_j \rangle x_j^* + \cdots + \langle x^*, x_n \rangle x_n^*, \]

\[ \langle x^*, x \rangle = \sum_{j=1}^{n} \langle x^*, x_j \rangle \langle x_j^*, x \rangle + \cdots + \langle x^*, x_n \rangle \langle x_n^*, x \rangle. \]

Let, now, $Y$ be an $m$-dimensional space with $1 \leq m < \infty$. Let $y_1, \ldots, y_m$ be an ordered basis for $Y$, and $y_1^*, \ldots, y_m^*$ be the corresponding adjoint basis for $Y^*$. If $T : X \rightarrow Y$ is linear, and we let

\[ t_{i,j} = \langle Ty_j^*, y_i \rangle, \quad i, j = 1, \ldots, n, \]
then we see that for \( j = 1, \ldots, n \),

\[
T_x^j = \langle T_x^j, y_1^* \rangle y_1 + \ldots + \langle T_x^j, y_m^* \rangle y_m
\]

\[
= \sum_{i=1}^{m} t_{i,j} y_i^* .
\]

Thus, for \( x \) in \( X \),

\[
T_x = \sum_{j=1}^{n} \langle x, x_j^* \rangle T_x^j
\]

\[
= \sum_{i=1}^{m} \left( \sum_{j=1}^{n} t_{i,j} \langle x, x_j^* \rangle \right) y_i^* .
\]

The operator \( T \) can be represented by the \( m \times n \) matrix \( A = [t_{i,j}] \), with respect to the bases \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_m \) of \( X \) and \( Y \) respectively, in the following sense:

\[
\begin{pmatrix}
  t_{1,1} & \cdots & t_{1,n} \\
  \vdots & \ddots & \vdots \\
  t_{m,1} & \cdots & t_{m,n}
\end{pmatrix}
\begin{pmatrix}
  \langle x, x_1^* \rangle \\
  \vdots \\
  \langle x, x_n^* \rangle
\end{pmatrix}
= 
\begin{pmatrix}
  \langle T_x, y_1^* \rangle \\
  \vdots \\
  \langle T_x, y_m^* \rangle
\end{pmatrix}.
\]

Now consider the adjoint operator \( T^* : Y^* \rightarrow X^* \). It can be easily seen that \( (X^*)^* \) can be identified with \( X \), and we can regard \( x_1, \ldots, x_n \) as the basis of \( (X^*)^* \) which is adjoint to the basis \( x_1^*, \ldots, x_n^* \) of \( X^* \). Since

\[
\langle T_x^*, x_i \rangle = \langle y_j, T_x^j \rangle = \langle T_x, y_j^* \rangle = \overline{t_{j,i}},
\]

for \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \), we see that the adjoint operator \( T^* \) is represented by the conjugate transpose matrix \( A^H = [\overline{t_{j,i}}] \), with respect to the bases \( y_1^*, \ldots, y_m^* \) and \( x_1^*, \ldots, x_n^* \) of \( Y^* \) and \( X^* \) respectively.
A commonly occurring situation is when $X = \mathbb{C}^n$, the set of all column vectors with $n$ entries of complex numbers. Let $e_j^{(n)}$ denote the column vector whose $i$-th entry $e_j^{(n)}(i)$ equals $\delta_{i,j}$. To save space, let $x = \begin{bmatrix} x(1) \\ \vdots \\ x(n) \end{bmatrix}$ in $\mathbb{C}^n$ be denoted by $[x(1), \ldots, x(n)]^t$, where the superscript $t$ denotes the transpose. Note that $x^H$ denotes the conjugate transpose of $x$, i.e., the row vector $[\overline{x(1)}, \ldots, \overline{x(n)}]$.

For $x \in \mathbb{C}^n$, we have

$$x = \sum_{j=1}^{n} x(j)e_j^{(n)}.$$  

so that $e_1^{(n)}, \ldots, e_n^{(n)}$ is a basis of $X$, the so-called standard basis. If $x^* \in X^*$ and we let

$$\langle x^*, e_j^{(n)} \rangle = y(j), \quad j = 1, \ldots, n,$$

then

$$\langle x^*, x \rangle = \sum_{j=1}^{n} \overline{x(j)} y(j).$$

so that $X^*$ can be identified again with the set $\mathbb{C}^n$ of column vectors $[y(1), \ldots, y(n)]^t$, and we can consider $x_j^* = e_j^{(n)}$, $j = 1, \ldots, n$, as the corresponding adjoint basis. Then we have for all $x \in X$ and $y \in X^*$,

$$\langle y, x \rangle = \sum_{j=1}^{n} \overline{x(j)} y(j) = x^H y.$$  

If $Y = \mathbb{C}^m$, and $T : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is linear, then

$$t_{i,j} = \langle Te_j^{(n)}, e_i^{(m)} \rangle = (e_i^{(m)})^H T e_j^{(n)}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
is simply the $i$-th entry of the $m$-vector $TE_j^{(n)}$ for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. Thus, $Tx$ is given by the product of the $m \times n$ matrix $[<TE_j^{(n)}, e_i^{(m)}>]$ with the $n \times 1$ matrix $x \in \ell^n$. Conversely, an $m \times n$ matrix defines a linear map from $\ell^n$ to $\ell^m$ in a natural way. We shall denote an operator and the corresponding matrix by the same letter $T$.

The $i$-th entry of the $n$-vector $T^*e_j^{(m)}$ is

$$<T^*e_j^{(m)}, e_i^{(n)}> = <e_i^{(n)}, Te_j^{(m)}> = \tau_{i,j}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Thus, the adjoint $T^*$ of an operator $T$ is given by the conjugate transpose $T^H$ of the corresponding matrix $T$.

(ii) Let $X = \ell^p$, $1 \leq p < \infty$, the space of all $p$-summable sequences of complex numbers, with the norm

$$\|x(1), x(2), \ldots \| = \left[ \sum_{j=1}^{\infty} |x(j)|^p \right]^{1/p},$$

for $x = [x(1), x(2), \ldots]_t$ in $X$. Then $X^*$ can be identified with $\ell^q$, where $1/p + 1/q = 1$, via the map $x^* \mapsto y$ with

$$<x^*, e_j> = y(j),$$

where $e_j = [0, \ldots, 0, 1, 0, \ldots]^t$, the entry 1 occurring only in the $j$-th place ([L], 13.4(b)). Now, for $x = \sum_{j=1}^{\infty} x(j)e_j$ in $X$ we have

$$<x^*, x> = \sum_{j=1}^{\infty} \overline{x(j)} y(j).$$

Let $T \in BL(\ell^p, \ell^q)$, and
Te_j = [t_1, j, t_2, j, \ldots]_t^t.

so that \langle Te_j, e_i \rangle = t_{i,j}. Since

$$Tx = \sum_{j=1}^{\infty} x(j)Te_j,$$

we have for \( i = 1,2,\ldots, \)

$$Tx(i) = \sum_{j=1}^{\infty} x(j)t_{i,j}.$$ 

Now, \( T^* \in BL(e^p, e^q), \) and

$$T^*e_j = [\bar{t}_{j,1}, \bar{t}_{j,2}, \ldots]_t^t,$$

since \( \langle T^*e_j, e_i \rangle = \langle e_j, Te_i \rangle = \bar{t}_{j,i}. \) We note that \( T \) and \( T^* \) are thus given by the infinite matrices \([t_{i,j}]\) and \([\bar{t}_{j,i}]\), \( i,j = 1,2,\ldots, \) respectively.

(iii) Let \( X = L^p([a,b]), \) \( 1 \leq p < \infty, \) the set of all \( p \)-integrable complex-valued functions on \([a,b]\) with the norm

$$\|x\|_p = \left( \int_a^b |x(t)|^p dm(t) \right)^{1/p},$$

where \( m \) is the Lebesgue measure. Then \( X^* \) can be identified with \( L^q([a,b]) \), where \( 1/p + 1/q = 1 \), since for every \( x^* \in X^* \), there is a unique \( y \in L^q([a,b]) \) such that

$$\langle x^*, x \rangle = \int_a^b x(t)y(t)dm(t), \quad x \in X.$$

(See [L], 14.3.)
Consider, for simplicity, \( p = 2 = q \), and let \( T \in \text{BL}(L^2([a,b])) \) be the integral operator

\[
T(x)(s) = \int_a^b k(s,t)x(t)\,dm(t), \quad x \in X,
\]

where \( \int_a^b \int_a^b |k(s,t)|^2 \,dm(s)\,dm(t) < \infty \). Then for all \( x,y \in X \), we have

\[
\langle T^*y, x \rangle = \langle y, Tx \rangle
\]

\[
= \int_a^b \overline{T(x)(t)y(t)}\,dm(t)
\]

\[
= \int_a^b \left( \int_a^b k(t,s) \overline{x(s)}\,dm(s) \right) y(t)\,dm(t)
\]

\[
= \int_a^b \overline{x(s)} \left( \int_a^b k(t,s) y(t)\,dm(t) \right)\,dm(s)
\]

so that for \( a \leq s \leq b \),

\[
T^*(s) = \int_a^b \overline{k(t,s)} y(t)\,dm(t).
\]

Thus, \( T^* \) is again an integral operator with kernel \( k^*(s,t) = \overline{k(t,s)} \).

(iv) Let \( X = C([a,b]) \), the set of all complex-valued continuous functions on the closed and bounded interval \([a,b]\) of the real line, with the supremum norm. Then for every \( x^* \in X^* \), there is a unique normalized function of bounded variation, say \( y \), such that

\[
\langle x^*, x \rangle = \int_a^b x(t)\,dy(t) \quad \text{for all} \quad x \in X.
\]

(See [L], 14.6).

Let \( T \) be an integral operator as in (iii) above, with \( k(s,t) \) continuous for \( s,t \in [a,b] \). Then for every \( x \in C([a,b]) \) and every normalized function \( y \) of bounded variation on \([a,b]\), we have, as earlier,
\[ \langle T^* y, x \rangle = \int_a^b x(s) \left( \int_a^b k(t,s) dy(t) \right) ds \]
\[ = \int_a^b x(s) dz(s) , \]

where
\[ z(s) = \int_a^s \left( \int_a^b k(t,u) dy(t) \right) du , \quad a \leq s \leq b . \]

Since this is true for every \( x \in X \), we see that for \( a \leq s \leq b \),
\[ T^* y(s) = z(s) = \int_a^s \left( \int_a^b k(t,u) du \right) dy(t) \]

Problems

1.1 Let \( Y \) be a closed subspace of \( X \), and \( x_0 \in X \) but \( x_0 \notin Y \). Then there is \( x^* \in X^* \) such that
\[ \langle x^*, y \rangle = 0 \quad \text{for all} \quad y \in Y , \quad \langle x^*, x_0 \rangle = 1 , \quad \text{and} \quad \| x^* \| = 1 / \text{dist}(x_0, Y) . \]

1.2 For fixed \( x \in X \), define \( f_x : X^* \rightarrow \mathbb{C} \) by \( f_x(x^*) = \langle x, x^* \rangle \). Then \( f_x \in X^{**} \). Identify \( x \) with \( f_x \), so that \( X \subseteq X^{**} \). Let \( E \subseteq X \). Then
\[ E^{\perp} \cap X = \text{the closure of span}(E) \text{ in } X . \]

If \( T \in \mathcal{BL}(X,Y) \), then
\[ Z(T) = X \cap R(T^*)^\perp , \]
\[ (1.10) \]
the closure of \( R(T) \) in \( X = X \cap Z(T^*)^\perp . \)
If $R(T)$ is closed, then

$$(1.11) \quad R(T^*) = Z(T)^\perp.$$ 

In general, does the closure of $R(T^*)$ in $X^*$ equal $Z(T)^\perp$?

1.3 Let $X$ and $Y$ be Hilbert spaces and $T \in BL(X,Y)$. Then the closure of $R(T)$ (resp., $R(T^*)$) equals $Z(T)^\perp$ (resp., $Z(T^*)^\perp$).

Also, $Z(T^*T) = Z(T)$ and the closure of $R(T^*T)$ equals the closure of $R(T^*)$. If $R(T)$ is closed, then $R(T^*)$ is closed and $R(T^*T) = R(T^*) = Z(T)^\perp$. Further, $T^*T$ is invertible if and only if $T$ is one to one and $R(T)$ is closed. (Hint: $R(T)$ is closed if and only if $v(T) = \inf \{ \|Tx\| : x \in Z(T)^\perp, \|x\| = 1 \} > 0$)

1.4 If $T \in BL(X,Y)$ is invertible, then $T^* \in BL(Y^*,X^*)$ is invertible and $(T^{-1})^* = (T^*)^{-1}$. The converse also holds. (See (8.1).)

1.5 Let $X = L^2([a,b])$ or $C([a,b])$, and

$$T^*x(s) = \int_a^b e^{st}x(t)dm(t), \ x \in X, \ a \leq s \leq b.$$ 

If $X = L^2([a,b])$, then $T^* = T$, while if $X = C([a,b])$, then

$$T^*y(s) = \int_a^b \frac{e^{ts}-e^{ta}}{t} dy(t)$$

for every normalized function $y$ of bounded variation on $[a,b]$; in particular, if $y \in C^1([a,b])$, then

$$T^*y(s) = \int_a^b \frac{e^{ts}-e^{ta}}{t} y'(t)dt.$$