As pointed out in Section 11, one approach to solving an eigenvalue problem for \( T \in BL(X) \) is to consider a nearby operator \( T_0 \in BL(X) \) which is simpler than \( T \), and first solve an eigenvalue problem for \( T_0 \). For example, if \( T \) is a large full matrix then \( T_0 \) can be a much smaller matrix, or a matrix with some special structure like tridiagonality or sparsity. We can attempt to refine the eigenelements \( \lambda_0 \) and \( \varphi_0 \) of \( T_0 \) for obtaining approximations of corresponding eigenelements \( \lambda \) and \( \varphi \) of \( T \). Several iterative procedures of this kind are given in Section 11 when \( \lambda_0 \) is a simple eigenvalue of \( T_0 \). In practice, one often chooses \( T_0 \) to be a bounded operator of finite rank. In the present section, we describe the step by step construction of the refinement schemes of Section 11 when \( T_0 \in BL(X) \) is of finite rank. The relevant algorithms can be implemented on a computer. (Many of the results of this section appear in the thesis [DE].)

Let us first study the spectrum of a bounded operator \( T_0 \) of finite rank. Since \( T_0 \) is a compact operator, one can appeal to the well known results for the spectra of compact operators. However, we prefer to give an independent treatment.

If the dimension of \( X \) is greater than the rank of \( T_0 \) (in particular, if \( X \) is infinite dimensional), then \( T_0 x = 0 \) for some nonzero \( x \in X \), i.e., \( 0 \) is an eigenvalue of \( T_0 \). Next, let \( \lambda_0 \neq 0 \). If \( \lambda_0 \) is not an eigenvalue of \( T_0 \), i.e., if \( T_0 - \lambda_0 I \) is one to one, then we show that \( T_0 - \lambda_0 I \) is also onto, so that \( \lambda_0 \) is not a spectral value of \( T_0 \). Let \( \tilde{T} = (T_0 - \lambda_0 I)|_{R(T_0)} \). Since \( R(T_0) \) is finite dimensional and \( \tilde{T} \) is one to one, we see that \( \tilde{T} \) maps \( R(T_0) \)
onto \( R(T_0) \). Let \( y \in X \). Since \( T_0y \in R(T_0) \), consider \( \tilde{x} \in R(T_0) \) such that \( (T_0 - \lambda_0 I)\tilde{x} = \tilde{T}_0 y \). If we let \( x = (\tilde{x} - y)/\lambda_0 \), we see that \( (T_0 - \lambda_0 I)x = (T_0y - T_0y + \lambda_0 y)/\lambda_0 = y \). Hence \( T_0 - \lambda_0 I \) is onto. Thus, every nonzero spectral value of \( T_0 \) is an eigenvalue of \( T_0 \).

In order to find the nonzero eigenvalues of \( T_0 \), we first set up some notations.

Consider \( x_1, \ldots, x_n \) in \( X \) and \( x_1^*, \ldots, x_n^* \) in \( X^* \). Then the map

\[
(17.1) \quad T_0^* x = \sum_{i=1}^n \langle x_i^*, x \rangle x_i^* , \quad x \in X .
\]

is a bounded operator on \( X \), and since \( R(T_0) \subseteq \text{span}\{x_1, \ldots, x_n\} \), it is of finite rank. Conversely, (3.8) shows that every bounded operator of finite rank is of this form. We shall assume throughout this section that \( T_0 \) is given by (17.1). Then it is easy to see that its adjoint \( T_0^* \) is given by

\[
(17.2) \quad T_0^* x^* = \sum_{i=1}^n \langle x_i^*, x \rangle x_i^* , \quad x^* \in X^* .
\]

Consider the matrix

\[
A = [a_{i,j}] , \quad a_{i,j} = \langle x_j, x_i^* \rangle , \quad i,j = 1, \ldots, n .
\]

Let \( \mathbb{C}^n \) denote the set of all column vectors \( \zeta = [c(1), \ldots, c(n)]^t \), \( c(i) \in \mathbb{C} \) for \( i = 1, \ldots, n \). As pointed out in Section 1, we denote the operator induced by the matrix \( A \) on \( \mathbb{C}^n \) also by \( A \).

Consider the linear map \( F : X \to \mathbb{C}^n \) given by

\[
(17.3) \quad Fx = [\langle x, x_1^* \rangle, \ldots, \langle x, x_n^* \rangle]^t , \quad x \in X ,
\]

and the linear map \( G : \mathbb{C}^n \to X \) given by

\[
(17.4) \quad G\zeta = c(1)x_1 + \ldots + c(n)x_n , \quad \zeta \in \mathbb{C}^n .
\]
Then

\[(17.5) \quad GF = T_0 \quad \text{and} \quad FG = A.\]

Of these, the first equality is immediate, and the second follows since for every $\zeta \in \mathbb{C}^n$,

\[
FG\zeta = c(1)Fx_1 + \ldots + c(n)Fx_n \\
= \sum_{i=1}^{n} c(i)[<x_1, x_1^*>, \ldots, <x_i, x_n^*>]^t \\
= [\sum_{i=1}^{n} c(i)<x_1, x_1^*>, \ldots, \sum_{i=1}^{n} c(i)<x_i, x_n^*>]^t \\
= [(A\zeta)(1), \ldots, (A\zeta)(n)]^t \\
= A\zeta.
\]

**Proposition 17.1** Let $0 \neq \lambda_0 \in \mathbb{C}$. Then $\lambda_0$ is an eigenvalue, a semisimple eigenvalue or a simple eigenvalue of $T_0$ if and only if it is an eigenvalue, a semisimple eigenvalue or a simple eigenvalue of $A$, respectively. If $u$ is an eigenvector of $A$ corresponding to $\lambda_0$, then $Gu$ is an eigenvector of $T_0$ corresponding to $\lambda_0$; if $x$ is an eigenvector of $T_0$ corresponding to $\lambda_0$, then $Fx$ is an eigenvector of $A$ corresponding to $\lambda_0$.

**Proof** For $k = 1, 2$, let

\[
V_k = Z((T_0 - \lambda_0 I)^k) \quad \text{and} \quad W_k = Z((A - \lambda_0 I)^k).
\]

We show that

\[
F(V_k) = W_k, \quad G(W_k) = V_k,
\]

and the maps $F|_{V_k}$, $G|_{W_k}$ are one to one.

It follows by (17.5) that for $x \in V_k$,

\[
(A - \lambda_0 I)^kFx = (FG - \lambda_0 I)^kFx = F(GF - \lambda_0 I)^kx = F(T_0 - \lambda_0 I)^kx = 0.
\]
so that $F_k \in W_k$. Similarly, for $\mathcal{C} \in W_k$,

$$(T_0 - \lambda_0 I)^k \mathcal{C} = (GF-\lambda_0 I)^k \mathcal{C} = G(FG-\lambda_0 I)^k \mathcal{C} = G(A-\lambda_0 I)^k \mathcal{C} = 0,$$

so that $\mathcal{C} \in V_k$. Thus, $F(V_k) \subset W_k$ and $G(W_k) \subset V_k$.

Next, let $x \in V_k$ and $Fx = 0$. Then

$$T_0^2 x = GFx = G(0) = 0,$$

so that $T_0^2 x = 0$ also. But since $x \in V_k$, we have $(T_0 - \lambda_0 I)^k x = 0$. This implies $\lambda_0^k x = 0$, i.e., $x = 0$, as $\lambda_0 \neq 0$. Thus, $F|_{V_k}$ is one to one. In an exactly similar manner, it follows that $G|_{W_k}$ is one to one. Now, since $W_k \subset \mathbb{C}^n$ is finite dimensional, we see that $W_k$ and $V_k$ have the same finite dimension and $F(V_k) = W_k$, $G(W_k) = V_k$ for $k = 1, 2$.

We immediately note that $\lambda_0$ is an eigenvalue of $T_0$ if and only if $V_1 \neq \{0\}$ if and only if $W_1 \neq \{0\}$ if and only if $\lambda_0$ is an eigenvalue of $A$. Since the $n$-dimensional operator $A$ has at most $n$ distinct eigenvalues, the same is true about the nonzero eigenvalues of $T_0$. As every nonzero spectral value of $T_0$ is an eigenvalue, we see that $\sigma(T_0)$ consists of at most $n + 1$ (isolated) points.

If $0 \neq \lambda_0 \in \sigma(T_0)$, then by considering a simple closed rectifiable curve $\Gamma_0$ which isolates $\lambda_0$ from 0 as well as the rest of $\sigma(T_0)$, we see that the range of the spectral projection $P_0$ associated with $T_0$ and $\lambda_0$ is contained in the range of $T_0$ (cf. Problem 6.7 and (7.19)). Since $T_0$ is of finite rank, it follows that $\lambda_0$ is an eigenvalue of $T_0$ of finite algebraic multiplicity. Since $A$ is itself a finite dimensional operator, it is obvious that every eigenvalue of $A$ has finite algebraic multiplicity.
Now, by Theorem 7.5(b) and Proposition 7.3, $\lambda_0$ is a semisimple eigenvalue of $T_0$ if and only if $V_2 = V_1 \neq \{0\}$ if and only if $W_2 = W_1 \neq \{0\}$ if and only if $\lambda_0$ is a semisimple eigenvalue of $A$
Also, $\lambda_0$ is a simple eigenvalue of $T_0$ if and only if $V_2 = V_1$ and $\dim V_1 = 1$, which is the case if and only if $W_2 = W_1$ and $\dim W_1 = 1$, i.e., $\lambda_0$ is a simple eigenvalue of $A$. The statements regarding the eigenvectors of $T_0$ and $A$ corresponding to $\lambda_0$ are also clear. //

COROLLARY 17.2 Let $0 \neq \lambda_0$ be a simple eigenvalue of $A$ and let $\psi$ be a corresponding eigenvector. Let $\chi$ be the eigenvector of $A^H$ corresponding to $\bar{\lambda}_0$ such that $\chi^H \psi = 1/\lambda_0$. Then $\lambda_0$ is a simple eigenvalue of $T_0$. Also,

$$\varphi_0 = u(1)x_1 + \ldots + u(n)x_n$$

and

$$\varphi_0^* = v(1)x_1^* + \ldots + v(n)x_n^*$$

are eigenvectors of $T_0$ and $T_0^*$ corresponding to $\lambda_0$ and $\bar{\lambda}_0$, respectively, and they satisfy $\langle \varphi_0, \varphi_0^* \rangle = 1$. In fact,

$$\langle \varphi_0, x_j^* \rangle = \lambda_0 u(j), \ j = 1, \ldots, n.$$  

The spectral projection $P_0$ associated with $T_0$ and $\lambda_0$ is given by

$$P_0x = \sum_{i,j=1}^n u(i)\overline{v(j)}\langle x, x_j^* \rangle x_i.$$  

Proof The spectral subspace associated with $A$ and $\lambda_0$ is one dimensional; if $\psi$ is a nonzero vector in this space, it follows by Theorem 8.3 that there is $\chi$ in the one dimensional spectral subspace associated with $A^* = A^H$ and $\bar{\lambda}_0$ such that $\langle \psi, \chi \rangle = \chi^H \psi = 1$. We can then let $\chi = \overline{\psi/\bar{\lambda}_0}$.
By Proposition 17.1, \( \lambda_0 \) is a simple eigenvalue of \( T_0 \) and 
\( \varphi_0 = G_{\varphi} = u_1 x_1 + \ldots + u_n x_n \) is a corresponding eigenvector of \( T_0 \).

Next, (17.2) shows that

\[ T_0^x = \sum_{i=1}^{n} \langle x_i^*, x_i^* \rangle x_i^* , \quad x^* \in X^* , \]

where \( \langle x_i^*, x_i^* \rangle \equiv \langle x_i, x_i \rangle \) for \( i = 1, \ldots, n \) and \( x^* \in X^* \). Also,

\[ \langle x_j^*, x_i^* \rangle = \langle x_i^*, x_j^* \rangle = \langle x_i, x_j \rangle = a_{i,j} . \]

Thus, we can apply Proposition 17.1 to \( T_0^x \), its simple eigenvalue \( \bar{\lambda}_0 \) and the operator \( A^x \). Hence \( \varphi_0^* = v_1 x_1^* + \ldots + v_n x_n^* \) is an eigenvector of \( T_0^x \) corresponding to \( \bar{\lambda}_0 \). Now, for \( j = 1, \ldots, n \),

\[ \langle \varphi_0, x_j^* \rangle = \sum_{i=1}^{n} u(i) x_i^* x_j^* = \sum_{i=1}^{n} \langle x_i, x_j^* \rangle u(i) = (A_{\varphi})(j) = \lambda_0 u(j) , \]

since \( A_{\varphi} = \lambda_0 \varphi \). This proves (17.8). Hence

\[ \langle \varphi_0, \varphi_0^* \rangle = \sum_{j=1}^{n} \langle \varphi_0, x_j^* \rangle v(j) = \lambda_0 \sum_{i=1}^{n} u(j) v(j) = 1 . \]

Finally, (17.9) follows from

\[ P_0 x = \langle x, \varphi_0^* \rangle \varphi_0 = \sum_{i=1}^{n} u(i) \langle x, \varphi_0^* \rangle x_i = \sum_{i,j=1}^{n} u(i) v(j) \langle x, x_j^* \rangle x_i . \]

**Remark 17.3** The converse of the above result also holds, i.e., if \( \lambda_0 \) is a simple eigenvalue of \( T_0 \), and \( \varphi_0 \) (resp., \( \varphi_0^* \)) is an eigenvector of \( T_0 \) (resp., \( T_0^x \)) corresponding to \( \lambda_0 \) (resp., \( \bar{\lambda}_0 \)) such that \( \langle \varphi_0, \varphi_0^* \rangle = 1 \), then \( \lambda_0 \) is a simple eigenvalue of \( A \), and

\[ u' = [\langle \varphi_0, x_1^* \rangle, \ldots, \langle \varphi_0, x_n^* \rangle]^t \quad \text{and} \quad v' = [\langle \varphi_0^*, x_1 \rangle, \ldots, \langle \varphi_0^*, x_n \rangle]^t \]

are eigenvectors of \( A \) and \( A^x \) corresponding to \( \lambda_0 \) and \( \bar{\lambda}_0 \).
respectively, such that
\[(y')^H u' = \sum_{i=1}^n \langle \varphi_0 \cdot x_i \rangle \langle \varphi_0 \cdot x_i \rangle = \sum_{i=1}^n \langle \varphi_0 \cdot x_i \rangle^* x_i \cdot \varphi_0 \rangle = \langle T_0 \varphi_0, \varphi_0 \rangle = \lambda_0 \varphi_0, \varphi_0 \rangle = \lambda_0 .\]

Let us recall the following iteration schemes for approximating eigenelements \( \lambda \) and \( \varphi \) of an operator \( T \in \text{BL}(X) \) which we have discussed in Section 11.

Let \( \lambda_0 \) be a simple eigenvalue of \( T_0 \in \text{BL}(X) \) with a corresponding eigenvector \( \varphi_0 \). Let \( \varphi_0^* \) be the eigenvector of \( T_0^* \) corresponding to \( \lambda_0 \) such that \( \langle \varphi_0, \varphi_0^* \rangle = 1 \). Let \( P_0 \) and \( S_0 \) denote the associated spectral projection and the reduced resolvent.

1. The Rayleigh-Schrödinger iteration scheme (11.18):
\[ \varphi_j = \varphi_{j-1} + S_0 \left[ -(T_0 - \lambda_1 I) \varphi_{j-1} + \sum_{k=2}^j \left( \lambda_k - \lambda_{k-1} \right) \varphi_{j-k} \right] \]

2. The fixed point iteration scheme (11.19):
\[ \varphi_j = \varphi_{j-1} + S_0 \left[ -T \varphi_{j-1} + \lambda_j \varphi_{j-1} \right] \]

3. The modified fixed point iteration scheme (11.31):
\[ \varphi_j = \frac{T \varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j} \left[ -T^2 \varphi_{j-1} + \frac{\langle T^2 \varphi_{j-1}, \varphi_0 \rangle}{\lambda_j} \right] \]

4. The Ahués iteration scheme (11.35):
\[ \varphi_j = \frac{T \varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j} \left[ -T^2 \varphi_{j-1} + \lambda_j T \varphi_{j-1} \right] . \]

Recall that in all these cases,
\[ \lambda_j = \langle T \varphi_{j-1}, \varphi_0 \rangle . \]
We remark that in the case of all the above iteration schemes,

\[ \langle \varphi_j, \varphi_0^* \rangle = \langle \varphi_0, \varphi_0^* \rangle = 1 \]

for \( j = 1, 2, \ldots \), as can be proved by induction on \( j \). Hence if we

let \( y_{j-1} \) equal \(-T(\lambda I)\varphi_{j-1} + \sum_{k=2}^{j} (\lambda_k - \lambda_{k-1})\varphi_{j-1} \), or

\(-T\varphi_{j-1} + y_{j-1} \), or \(-T^2\varphi_{j-1} + \langle T^2\varphi_{j-1}, \varphi_0^* \rangle \varphi_{j-1}/\lambda_j \), then

\[ \langle y_{j-1}, \varphi_0^* \rangle = 0 \] i.e., \( P_0 y_{j-1} = 0 \). Thus, to implement the first three schemes, we need to calculate

(i) \( \langle Tx, \varphi_0^* \rangle \) for various \( x \in X \) and

(ii) \( S_0 y \) for various \( y \in X \) which satisfy \( P_0 y = 0 \).

We shall comment on the implementation of the fourth scheme in Remark (17.7).

We have already seen in Corollary 17.2 how to find \( \lambda_0, \varphi_0 \) and \( \varphi_0^* \) in case \( T_0 \) is of finite rank. In the next result we give a procedure for finding \( S_0 y \), where \( y \in X \) and \( P_0 y = 0 \).

**PROPOSITION 17.4** Let \( 0 \neq \lambda_0, y, \varphi_0, \chi \) and \( \varphi_0^* \) be as in Corollary 17.2. For \( y \in X \) with \( P_0 y = 0 \), the unique solution \( x \) of the system

\[ (17.9) \quad (T_0 - \lambda_0 I)x = y, \quad P_0 x = 0 \]

is given by

\[ (17.10) \quad x = S_0 y = \frac{1}{\lambda_0} \left[ -y + \sum_{i=1}^{n} \alpha(i)x_i \right] \]

where \( \alpha = [\alpha(1), \ldots, \alpha(n)]^t \) is the unique solution of

\[ (17.11) \quad (A - \lambda_0 I)\alpha = [\langle y, x_1^* \rangle, \ldots, \langle y, x_n^* \rangle]^t, \quad \chi^H \alpha = 0. \]
Also, we have

(17.12) \[ \langle x, x_i^* \rangle = \alpha(i), \quad i = 1, \ldots, n. \]

**Proof** Since \( S_0 | (I - P_0)X \) is the inverse of \( (T_0 - \lambda_0 I) | (I - P_0)X \), it is clear that \( x = S_0 y \) is the unique solution of (17.9). Now, we have

\[ x = \frac{1}{\lambda_0} (-y + T_0 x) = \frac{1}{\lambda_0} (-y + GFx). \]

Thus, to obtain \( x \), it is enough to find \( Fx \). Now,

\[ Fx = [\langle x, x_1^* \rangle, \ldots, \langle x, x_n^* \rangle]^t, \]

\[ \chi^H Fx = \sum_{i=1}^n \langle x, x_i^* \rangle \chi(i) = \langle x, \sum_{i=1}^n \chi(i)x_i^* \rangle = \langle x, \varphi_0^* \rangle = 0, \]

since \( P_0 x = 0 \). Similarly, \( Fy = [\langle y, x_1^* \rangle, \ldots, \langle y, x_n^* \rangle]^t \) and \( \chi^H Fy = 0 \), since \( P_0 y = 0 \). Also,

\[ (A - \lambda_0 I)Fx = (FG - \lambda_0 I)Fx = F(GF - \lambda_0 I)x = F(T_0 - \lambda_0 I)x = Fy. \]

Since \( \lambda_0 \) is a simple eigenvalue of \( A \) and \( \chi \) is an eigenvector of \( A^* \) corresponding to \( \lambda_0^* \), we see that whenever \( \chi^H \beta = 0 \), the system

\[ (A - \lambda_0 I)\alpha = \beta, \quad \chi^H \alpha = 0 \]

has a unique solution in \( \mathbb{C}^n \). We have just now seen that if \( \beta = Fy \), then \( \alpha = Fx \) is such a solution. In particular, for \( i = 1, \ldots, n \), we have \( \alpha(i) = Fx(i) = \langle x, x_i^* \rangle \). This establishes (17.12) and completes the proof. //

We are now in a position to find the iterates \( \lambda_j \) and \( \varphi_j \) for a variety of iteration schemes. The problem of solving the operator equations \( (T_0 - \lambda_0 I)x = y \), \( P_0 x = 0 \) (where \( P_0 y = 0 \)) is reduced to solving matrix equations in the following manner. A similar problem is considered in [WH] when \( \lambda_0 \in \rho(T_0) \).
Theorem 17.5 Let \( 0 \neq \lambda_0, \mu, \nu \), and \( \varphi_0, \chi \) be as in Corollary 17.2. Consider an iteration scheme

\[
\begin{align*}
\lambda_j &= \langle T \varphi_{j-1}, \varphi_0 \rangle, \\
\varphi_j &= \xi_j + S_0 \eta_j,
\end{align*}
\]

where \( \xi_j, \eta_j \in X \) may depend on \( T, \lambda_0, \ldots, \lambda_j, \nu_0, \nu_1, \ldots, \nu_{j-1} \), and where \( P_0 \eta_j = 0 \). Then for \( j = 1, 2, \ldots \)

\[
\begin{align*}
\lambda_j &= \sum_{i=1}^{n} \langle T \varphi_{j-1}, x_i^* \rangle \nu(i), \\
\varphi_j &= \xi_j + \frac{1}{\lambda_0} \left[-\eta_j + \sum_{i=1}^{n} \alpha_j(i)x_i \right],
\end{align*}
\]

where \( \alpha_j = [\alpha_j(1), \ldots, \alpha_j(n)]^t \) is the unique solution of

\[
(A - \lambda_0 I) \alpha_j = \beta_j, \quad \chi^H \alpha_j = 0,
\]

with

\[
\beta_j = [\langle \eta_j, x_1^* \rangle, \ldots, \langle \eta_j, x_n^* \rangle]^t.
\]

Moreover, we have for \( j = 1, 2, \ldots \) and \( i = 1, \ldots, n \),

\[
\langle \varphi_j, x_i^* \rangle = \langle \xi_j, x_i^* \rangle + \alpha_j(i).
\]

Proof Since \( \varphi_0 = \sum_{i=1}^{n} \nu(i)x_i^* \), the expression (17.14) for \( \lambda_j \) follows immediately. To prove (17.15), we note that by Proposition 17.4,

\[
S_0 \eta_j = \frac{1}{\lambda_0} \left[-\eta_j + \sum_{i=1}^{n} \alpha_j(i)x_i \right],
\]

where \( \alpha_j = [\alpha_j(1), \ldots, \alpha_j(n)]^t \) satisfies

\[
(A - \lambda_0 I) \alpha_j = \beta_j, \quad \chi^H \alpha_j = 0,
\]

with \( \beta_j = [\langle \eta_j, x_1^* \rangle, \ldots, \langle \eta_j, x_n^* \rangle]^t \). Hence (17.15) holds. Lastly, by
letting $x = S^0_0 \eta_j$ in (17.12), we see that $\langle S^0_0 \eta_j, \text{x}_i^* \rangle = \alpha_j(1)$. Hence the relation (17.16) follows. //

**Remark 17.6** The above theorem can be applied to the Rayleigh-Schrödinger iteration scheme (11.18) with

$$\xi_j = \varphi_{j-1}, \quad \eta_j = -(T-\lambda_1 I)\varphi_{j-1} + \sum_{k=2}^{j} (\lambda_k - \lambda_{k-1}) \varphi_{j-k}$$

and to the fixed point iteration scheme (11.19) with

$$\xi_j = \varphi_{j-1}, \quad \eta_j = -T\varphi_{j-1} + \lambda_j \varphi_{j-1}.$$ 

In both the cases, we have for $j = 1, 2, \ldots$ and $i = 1, \ldots, n$,

$$\langle \varphi_j, \text{x}_i^* \rangle = \langle \varphi_{j-1}, \text{x}_i^* \rangle + \alpha_j(1)$$

$$= \langle \varphi_0, \text{x}_i^* \rangle + \alpha_1(1) + \ldots + \alpha_j(1)$$

$$= \lambda_0 u(i) + \alpha_1(1) + \ldots + \alpha_j(1),$$

because of (17.16) and (17.8). If we let

$$\alpha_0(i) = \lambda_0 u(i), \quad i = 1, \ldots, n,$$

then we have

$$\langle \varphi_j, \text{x}_1^* \rangle = \sum_{k=0}^{j-1} \alpha_k(1), \quad j = 0, 1, 2, \ldots.$$ 

This relation can be used in calculating the right hand sides

$$\beta_j = [\langle \eta_j, \text{x}_1^* \rangle, \ldots, \langle \eta_j, \text{x}_n^* \rangle]^t, \quad j = 1, 2, \ldots, as follows. For the iteration scheme (11.18), we have

$$\beta_j(1) = \langle \eta_j, \text{x}_1^* \rangle$$

$$= -\langle T\varphi_{j-1}, \text{x}_1^* \rangle + \sum_{k=0}^{j-1} \frac{1}{\lambda_{p}} \lambda_{j-k} \alpha_k(1)$$

$$= -\langle T\varphi_{j-1}, \text{x}_1^* \rangle + \sum_{k=0}^{j-1} \lambda_{j-k} \alpha_k(1).$$
For the iteration scheme (11.19), we have

$$(17.19) \quad \beta_j(i) = \langle \eta_j, x_i \rangle = -\langle T\varphi_{j-1}, x_i \rangle + \lambda_j \sum_{k=0}^{j-1} \alpha_k(i).$$

We can also apply Theorem 17.5 to the modified fixed point iteration scheme (11.31) with

$$\xi_j = \frac{T\varphi_{j-1}}{\lambda_j}, \quad \eta_j = \frac{1}{\lambda_j} \left[ -\langle T^2\varphi_{j-1}, x_1 \rangle + \frac{\mu_j}{\lambda_j} \langle T\varphi_{j-1}, x_1 \rangle \right].$$

In this case, we have

$$(17.20) \quad \beta_j(i) = \langle \eta_j, x_i \rangle = \frac{1}{\lambda_j} \left[ -\langle T^2\varphi_{j-1}, x_1 \rangle + \frac{\mu_j}{\lambda_j} \langle T\varphi_{j-1}, x_1 \rangle \right],$$

where

$$(17.21) \quad \mu_j = \langle T^2\varphi_{j-1}, \varphi_0 \rangle = \sum_{i=1}^{n} \langle T^2\varphi_{j-1}, x_i \rangle \nu(1).$$

Thus, we also need to calculate $T^2\varphi_{j-1}$ and $\langle T^2\varphi_{j-1}, x_1 \rangle$.

**Remark 17.7** In case we have an iteration scheme

$$\lambda_j = \langle T\varphi_{j-1}, \varphi_0 \rangle, \quad \varphi_j = \xi_j + S_0\tilde{\eta}_j, \quad j = 1, 2, \ldots,$$

where $\tilde{\eta}_j$ may not satisfy $P_0\tilde{\eta}_j = 0$, we can let

$$\eta_j = \tilde{\eta}_j - P_0\tilde{\eta}_j,$$

and observe that since $S_0 P_0 = 0$, we have

$$\varphi_j = \xi_j + S_0\eta_j, \quad P_0\eta_j = 0.$$

We can then apply Theorem 17.5. This is the situation for Ahue's iteration scheme (11.35), where

$$\varphi_j = \frac{T\varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j} \left[ -T^2\varphi_{j-1} + \lambda_j T\varphi_{j-1} \right], \quad j = 1, 2, \ldots.$$
Let \( \xi_j = \frac{T\phi_{j-1}}{\lambda_j} \) and \( \tilde{\eta}_j = \frac{1}{\lambda_j} \left[ -T^2\phi_{j-1} + \lambda_j T\phi_{j-1} \right] \). Then

\[
(17.22) \quad \eta_j = \frac{1}{\lambda_j} \left[ -T^2\phi_{j-1} + \lambda_j T\phi_{j-1} \right] - \frac{1}{\lambda_j} \left[ \langle T^2\phi_{j-1}, \phi_0 \rangle + \lambda_j \langle T\phi_{j-1}, \phi_0 \rangle \right] \phi_0
\]

\[
= \frac{1}{\lambda_j} \left[ -T^2\phi_{j-1} + \lambda_j T\phi_{j-1} + (\mu_j - \lambda_j^2)\phi_0 \right],
\]

where \( \mu_j \) is given by (17.21). In this case, we have

\[
(17.23) \quad \beta_j(i) = \langle \eta_j, x_i^* \rangle
\]

\[
= \frac{1}{\lambda_j} \left[ -\langle T^2\phi_{j-1}, x_i^* \rangle + \lambda_j \langle T\phi_{j-1}, x_i^* \rangle + \lambda_0 (\mu_j - \lambda_j^2)u(i) \right].
\]

We now write down algorithms for implementing various iteration schemes considered so far; the iterates approximate eigenelements of a bounded operator \( T \) on \( X \). Let

\[ T_0 x = \sum_{i=1}^{n} \langle x, x_i^* \rangle x_i, \quad x \in X. \]

be a fixed finite rank operator on \( X \), and consider the matrix

\[ A = [\langle x_j, x_i^* \rangle], \quad i, j = 1, \ldots, n. \]

**Algorithm 17.8** For \( j = 0, 1, 2, \ldots \), the iterates

\[ \lambda_j = \langle T\phi_{j-1}, \phi_0^* \rangle, \]

\[ \phi_j = \phi_{j-1} + S_0 \left[ -(T - \lambda_1 I)\phi_{j-1} + \sum_{k=2}^{j} (\lambda_k - \lambda_{k-1})\phi_{j-k} \right] \]

of the Rayleigh–Schrödinger scheme (11.18) can be found as follows.

**Step 1**

(i) Solve the eigenvalue problem for \( A \). If \( \lambda_0 \) is a nonzero simple eigenvalue of \( A \), find a corresponding eigenvector \( \bar{u} = [u(1), \ldots, u(n)]^t \).

(ii) Find the eigenvector \( \bar{v} = [v(1), \ldots, v(n)]^t \) of \( A^H \) corresponding to \( \bar{\lambda}_0 \) such that \( \bar{v}^H \bar{u} = \frac{1}{\lambda_0} \).
Step 2  Put \( \alpha_0 = \lambda_0 u \), \( \varphi_0 = u(1)x_1 + \ldots + u(n)x_n \), and for \( j = 1, 2, \ldots \), repeat the following:

(i) Calculate \( \langle T\varphi_{j-1}, x^*_1 \rangle \), \( i = 1, \ldots, n \), and put
\[
\lambda_j = \sum_{i=1}^{n} \langle T\varphi_{j-1}, x^*_1 \rangle v(i).
\]

(ii) Let \( \beta_j = [\beta_j(1), \ldots, \beta_j(n)]^t \) with
\[
\beta_j(i) = -\langle T\varphi_{j-1}, x^*_1 \rangle + \sum_{k=0}^{j-1} \lambda_j^{j-k} \alpha_k(i), \quad i = 1, \ldots, n.
\]
Find \( \alpha_j = [\alpha_j(1), \ldots, \alpha_j(n)]^t \) such that
\[
\mathcal{H} \alpha_j = 0, \quad (A-\lambda_0 I)\alpha_j = \beta_j.
\]

(iii) Calculate \( T\varphi_{j-1} \) and put
\[
\varphi_j = \frac{1}{\lambda_0} \left[ \sum_{i=1}^{n} \alpha_j(i)x_1 + \sum_{k=1}^{j} (\lambda_k - \lambda_j)\varphi_{j-k} + T\varphi_{j-1} \right].
\]

**Algorithm 17.9** For \( j = 0, 1, 2, \ldots \), the iterates
\[
\lambda_j = \langle T\varphi_{j-1}, \varphi_0^* \rangle,
\]
\[
\varphi_j = \varphi_{j-1} + S_0 \left[ -T\varphi_{j-1} + \lambda_j \varphi_{j-1} \right]
\]
of the fixed point scheme (11.19) can be found as in Step 1 and Step 2 of Algorithm 17.8 except for the following changes:

In Step 2 (ii), for \( i = 1, \ldots, n \), let
\[
\beta_j(i) = -\langle T\varphi_{j-1}, x^*_1 \rangle + \sum_{k=0}^{j-1} \lambda_j^{j-k} \alpha_k(i)
\]
and in Step 2 (iii), let
\[
\varphi_j = \frac{1}{\lambda_0} \left[ \sum_{i=1}^{n} \alpha_j(i)x_1 + (\lambda_0 - \lambda_j)\varphi_{j-1} + T\varphi_{j-1} \right].
\]

**Algorithm 17.10** For \( j = 0, 1, 2, \ldots \), the iterates
\[
\lambda_j = \langle T\varphi_{j-1}, \varphi_0^* \rangle,
\]
\[
\varphi_j = \left[ \frac{T \varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j} \left[ -T^2 \varphi_{j-1} + \frac{\langle T^2 \varphi_{j-1}, \varphi_0^* \rangle}{\lambda_j} T\varphi_{j-1} \right] \right].
\]
of the modified fixed point scheme (11.31) can be found as in Step 1 and Step 2 of Algorithm 17.8 except for the following changes:

In Step 2 (i), calculate additionally $\langle T^2 \varphi_{j-1}, x^*_i \rangle$, $i = 1, \ldots, n$, and put

$$\mu_j = \frac{1}{\lambda_j} \sum_{i=1}^{n} \langle T^2 \varphi_{j-1}, x^*_i \rangle v(1).$$

In Step 2 (ii), for $i = 1, \ldots, n$, let

$$\beta_j(i) = \frac{1}{\lambda_j} \left[ -\langle T^2 \varphi_{j-1}, x^*_i \rangle + \mu_j \langle T \varphi_{j-1}, x^*_i \rangle \right].$$

In Step 2 (iii), calculate additionally $T^2 \varphi_{j-1}$, and put

$$\varphi_j = \frac{1}{\lambda_0 \lambda_j} \left[ \lambda_j \sum_{i=1}^{n} a_j(i) x_i + \left( \lambda_0 - \frac{\mu_j}{\lambda_j} \right) T \varphi_{j-1} + T^2 \varphi_{j-1} \right].$$

**ALGORITHM 17.11** For $j = 0, 1, 2, \ldots$, the iterates

$$\lambda_j = \langle T \varphi_{j-1}, \varphi^*_0 \rangle,$$

$$\varphi_j = \frac{T \varphi_{j-1}}{\lambda_j} + \frac{S_0}{\lambda_j} [-T^2 \varphi_{j-1} + \lambda_j T \varphi_{j-1}]$$

of the Ahués scheme (11.35) can be found as in Step 1 and Step 2 of Algorithm 17.8 except for the following changes:

In Step 2 (i), calculate additionally $\langle T^2 \varphi_{j-1}, x^*_i \rangle$, $i = 1, \ldots, n$, and put

$$\mu_j = \frac{1}{\lambda_j} \sum_{i=1}^{n} \langle T^2 \varphi_{j-1}, x^*_i \rangle v(1).$$

In Step 2 (ii), for $i = 1, \ldots, n$, let

$$\beta_j(i) = \frac{1}{\lambda_j} \left[ -\langle T^2 \varphi_{j-1}, x^*_i \rangle + \lambda_j \langle T \varphi_{j-1}, x^*_i \rangle + \lambda_0 (\mu_j - \lambda_j^2) u(1) \right].$$
In Step 2 (iii), calculate additionally $T^2\varphi_{j-1}$ and put

$$\varphi_j = \frac{1}{\lambda_0\lambda_j} \left[ \lambda_j \sum_{i=1}^{n} \alpha_j(i)x_i + (\lambda_j^2 - \mu_j)\varphi_0 + (\lambda_0 - \lambda_j)T\varphi_{j-1} + T^2\varphi_{j-1} \right].$$

In writing the above algorithms, we have made use of Theorem 17.3 and the expressions (17.18), (17.19), (17.20) and (17.23) for $\beta_j(i), i = 1, \ldots, n$, $j = 1, 2, \ldots$.

**REMARK 17.12** We now make some remarks regarding the choice of a finite rank operator $T_0$ and its simple nonzero eigenvalue $\lambda_0$. We have seen in Section 14 that if $\lambda$ is a simple nonzero eigenvalue of $T \in BL(X)$, separated by a simple closed rectifiable curve $\Gamma$ from zero as well as from the rest of the spectrum of $T$, and if $(T_n)$ is a resolvent operator approximation of $T$, then for all large $n$, $T_n$ has a simple nonzero eigenvalue $\lambda_n$ inside $\Gamma$, and it is the only spectral value of $T_n$ inside $\Gamma$. Let $\varphi_n$ (resp., $\varphi^*_n$) be an eigenvector of $T_n$ (resp., $T^*_n$) corresponding to $\lambda_n$ (resp., $\lambda^*_n$) such that $\|\varphi_n\| = 1 = \|\varphi^*_n\|$. Then if $n_0$ is sufficiently large, and we make the choice $T_0 = T_{n_0}$, $\lambda_0 = \lambda_{n_0}$, $\varphi_0 = \varphi_{n_0}$, $\varphi^*_0 = \varphi^*_{n_0}$, all the iteration schemes considered in Section 11 converge and yield the eigenvalue $\lambda$ and a corresponding eigenvector $\varphi$ of $T$ which satisfies $\langle \varphi, \varphi^*_0 \rangle = 1$. Moreover, if $\lambda$ is the dominant spectral value of $T$, then for all large $n$, $\lambda_n$ is the dominant spectral value of $T_n$; if $\lambda$ has the second largest absolute value among the elements of $\sigma(T)$, then the same holds for $\lambda_n$ as far as the elements of $\sigma(T_n)$ are concerned. Hence the choice of $\lambda_{n_0}$ from among the elements of $\sigma(T_{n_0})$ is dictated by which eigenvalue $\lambda$ of $T$ we wish to approximate.

Thus, $T_0$ can be chosen to be a member $T_{n_0}$ of a sequence $(T_n)$ of finite rank operators, which is a resolvent operator approximation of
If $T_n$ is of rank $n$, then one attempts to keep $n_0$ small, since in the first step of the algorithms, we need to find the eigenvalues of an $n_0 \times n_0$ matrix. However, there is no practically verifiable criterion for deciding the smallest integer $n_0$ which implies convergence of an iteration scheme, and one has to proceed by a trial and error method. The numerical experiments in Section 19 show that even $n_0 = 2$ works in some cases; in general, though, it is safer to choose $n_0 \geq 4$. The norm approximation and the collectively compact approximation provide useful examples of resolvent operator approximation. We list below some important choices for $T_0$ in these categories.

Let $\pi_0$ be a bounded projection of finite rank given by

$$\pi_0 x = \sum_{i=1}^{n} \langle x, e_i^* e_i \rangle,$$

where $e_i \in X$, $e_i^* \in X^*$ with $\langle e_j, e_i^* x \rangle = \delta_{i,j}$ for $i, j = 1, \ldots, n$.

Let (cf. (15.1))

$$T_0^p = \pi_0 T, \quad T_0^s = T \pi_0, \quad T_0^g = \pi_0 T \pi_0.$$

In case $T$ is a Fredholm integral operator

$$T x(s) = \int_a^b k(s, t)x(t)dt, \quad x \in X \text{ and } s \in [a, b],$$

where $X = L^2([a, b])$ or $C([a, b])$, let (cf. (16.3))

$$T_0^d = \sum_{i=1}^{n} \left[ \int_a^b y_i(t) s(t) dt \right] x_i, \quad x \in X,$$

where $x_i, y_i \in X$, $i, j = 1, \ldots, n$; also for $X = C([a, b])$ and a quadrature formula
\[ f_0(x) = \sum_{i=1}^{n} w_i x(t_i), \quad x \in X, \]

with \( w_i \in \mathbb{C} \), and \( a \leq t_1 < t_2 < \ldots < t_n \leq b \), let (cf. (16.6))

\[ T_0^N x(s) = \sum_{i=1}^{n} w_i k(s, t_i) x(t_i); \]

if the projection \( \pi_0 \) is an interpolatory projection with nodes at \( t_i \), \( i = 1, \ldots, n \), i.e., \( \langle x, e_i^* \rangle = x(t_i) \), so that

\[ \pi_0 x = \sum_{i=1}^{n} x(t_i) e_i, \quad x \in X, \]

with \( e_j(t_i) = \delta_{i,j} \), then let (cf. (16.7))

\[ T_0^F x = \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} w_{ij} k(t_i, t_j) x(t_j) \right] e_i. \]

The expressions for \( x_i \) and \( x_i^* \) appearing in

\[ T_0 x = \sum_{i=1}^{n} \langle x, x_i^* \rangle x_i, \quad x \in X, \]

for the above choices for \( T_0 \) as well as the matrix \( A = [\langle x_j, x_i^* \rangle] \) are tabulated below.
\begin{align*}
T_0 & \quad x_1 \quad \langle x, x_1^* \rangle \\
\downarrow & \quad \downarrow \\
T_0^p & \quad e_1 \quad \langle x, T e_1^* \rangle \\
\downarrow & \quad \downarrow \\
T_0^s & \quad T e_1 \quad \langle x, e_1^* \rangle \\
\downarrow & \quad \downarrow \\
T_0^c & \begin{cases} 
1 & e_i \quad \sum_{j=1}^n \langle x, e_j^* \rangle < e_j, T e_i^* > \quad \langle < T e_j, e_i^* > \\
2 & \sum_{j=1}^n \langle T e_i, e_j^* \rangle e_j \quad \langle x, e_1^* \rangle \quad \langle < T e_j, e_i^* > 
\end{cases}
\downarrow & \quad \downarrow \\
T_0^d & \quad x_1 \quad \int_a^by_i(t)x(t)dt \quad \int_a^by_i(t)x(t)dt \\
\downarrow & \quad \downarrow \\
T_0^n & \quad w_i k(\cdot, t_1) \quad x(t_1) \quad [w_j k(t_1, t_j)] \\
\downarrow & \quad \downarrow \\
T_0^r & \begin{cases} 
1 & e_i \quad \sum_{j=1}^n w_j k(t_1, t_j) x(t_j) \quad [w_j k(t_1, t_j)] \\
2 & \sum_{j=1}^n w_i k(t_1, t_j) e_j \quad x(t_1) \quad [w_j k(t_1, t_j)] 
\end{cases}
\end{align*}

Table 17.1
We conclude this section by pointing out some interesting relationships among the iterates for the projection, Sloan and Galerkin methods for approximating \( T \in BL(X) \).

**Proposition 17.13** Consider a projection \( \pi_0 \in BL(X) \). Let \( \lambda_0 \) be a nonzero simple eigenvalue of \( T_0^G = \pi_0 T \pi_0 \), and let \( \varphi_0 \) (resp., \( \varphi_0^* \)) be an eigenvector of \( T_0^G \) (resp., \( (T_0^G)^* \)) corresponding to \( \lambda_0 \) (resp., \( \bar{\lambda}_0 \)) such that \( \langle \varphi_0, \varphi_0^* \rangle = 1 \). Then

(a) \( \lambda_0 \) is a simple eigenvalue of \( T_0^P = \pi_0 T \), and of \( T_0^S = T \pi_0 \); the elements

\[
\begin{align*}
\varphi_0^P &= \varphi_0, \\
\varphi_0^* &= T^* \varphi_0^* / \lambda_0, \\
\varphi_0^S &= T \varphi_0 / \lambda_0, \\
\varphi_0^{*S} &= \varphi_0^*
\end{align*}
\]

are eigenvectors of \( T_0^P \), \( (T_0^P)^* \), \( T_0^S \), \( (T_0^S)^* \) corresponding to \( \lambda_0 \), \( \bar{\lambda}_0 \), \( \lambda_0 \), \( \bar{\lambda}_0 \), respectively, such that

\[
\langle \varphi_0, \varphi_0^* \rangle = 1 = \langle \varphi_0^*, \varphi_0^* \rangle.
\]

(b) For \( j = 1, 2, \ldots \), let

\[
\lambda_j^P, \varphi_j^P \text{ and } \lambda_j^S, \varphi_j^S
\]

denote the corresponding iterates in any one of the iteration schemes (11.18), (11.19), (11.31) or (11.35). Then for \( j = 0, 1, 2, \ldots \),

\[
\lambda_j^S = \lambda_j^P \text{ and } \varphi_j^S = \frac{1}{\lambda_0} T \varphi_j^P.
\]

**Proof** We note from Table 17.1 that for all three operators \( T_0^G, T_0^P \) and \( T_0^S \), the corresponding matrix \( A = [\langle x_j, x_i^* \rangle] \) is the same, namely, \([\langle T e_j, e_i^* \rangle], \ i, j = 1, \ldots, n \). Hence it follows from Remark 17.3 and Corollary 17.2 that \( \lambda_0 \) is a simple eigenvalue of \( A \), and hence of \( T_0^P \) and \( T_0^S \).
Since \( \pi_0^T \varphi_0 = \lambda_0 \varphi_0 \) and \( \pi_0^* T_0^G \varphi_0 = \bar{\lambda}_0 \varphi_0 \), we see that \( \varphi_0 \in \pi_0^* X \) and \( \varphi_0 \in \pi_0^* X^* \), i.e., \( \pi_0^* \varphi_0 = \varphi_0 \) and \( \pi_0^* \varphi_0 = \varphi_0 \). Now,

\[
T_0^P \varphi_0 = \pi_0 T \varphi_0 = \pi_0 \pi_0^* T_0^G \varphi_0 = T_0^G \varphi_0 = \lambda_0 \varphi_0 ,
\]

\[
(T_0^P)^* (T_0^G \varphi_0) = T_0^G \pi_0^* (T_0^P \pi_0^* \varphi_0) = T_0^G \pi_0^* \varphi_0 = \bar{\lambda}_0 \varphi_0 ,
\]

\[
\langle \varphi_0, T_0^P \varphi_0 \rangle = \langle \pi_0 \pi_0^* T_0^P \pi_0^* \varphi_0 \rangle = \langle \varphi_0, \pi_0 T_0^G \varphi_0 \rangle = \lambda_0 \lambda_0 = \lambda_0 ,
\]

\[
T_0^S(T_0^G \varphi_0) = T_0^P(T_0^G \varphi_0) = T_0^G \lambda_0 \varphi_0 ,
\]

\[
(T_0^S)^* (T_0^G \varphi_0) = \pi_0 \pi_0^* (T_0^S \pi_0^* \varphi_0) = (T_0^G)^* \varphi_0 = \bar{\lambda}_0 \bar{\lambda}_0 ,
\]

\[
\langle T_0^S, T_0^G \varphi_0 \rangle = \langle \varphi_0, T_0^S \varphi_0 \rangle = \lambda_0 .
\]

Hence the results in part (a) follow.

(b) Let \( P_0^P \) (resp., \( P_0^S \)) denote the spectral projection associated with \( T_0^P \) (resp., \( T_0^S \)) and \( \lambda_0 \). Then for \( x \in X \),

\[
P_0^P x = \langle x, T_0^P \varphi_0 \rangle \varphi_0 / \lambda_0 = \langle T_0^P x, \varphi_0 \rangle \varphi_0 / \lambda_0 .
\]

(17.26)

\[
P_0^S x = \langle x, T_0^S \varphi_0 \rangle \varphi_0 / \lambda_0 .
\]

Hence \( P_0^S T_0 = T_0^P P_0^S \). Next, we show that

(17.27)

\[
S_0^T = T_0^S P_0^S ,
\]

where \( S_0^P \) (resp., \( S_0^S \)) is the reduced resolvent associated with \( T_0^P \) (resp., \( T_0^S \)) and \( \lambda_0 \). First, let \( x \in X \) with \( P_0^P x = 0 \). Then by (17.26), \( P_0^S T_0 x = 0 \), and since

\[
T_0^S = T_0^P ,
\]

we have

\[
(T_0^S - \lambda_0 I) S_0^S x = T(T_0^P - \lambda_0 I) P_0^P x = T(I - P_0^P) x = T x .
\]

Thus, \( y = T_0^S x \) is the unique element of \( X \) which satisfies

\[
(S_0^S - \lambda_0 I)y = Tx , \quad P_0^S y = 0 ,
\]
where \( P_0^S Tx = 0 \). Hence \( y = S_0^S Tx \), i.e., \( T S_0^P x = S_0^S Tx \). Next, if \( x \in X \) with \( P_0^P x = x \), then again by (17.26), we see that \( P_0^S Tx = Tx \).

Hence

\[
T S_0^P x = T S_0^P P x = 0 = S_0^P S_0^P Tx = S_0^S Tx .
\]

Thus, (17.27) holds by considering \( x = P_0^P x + (I-P_0^P)x \) for all \( x \in X \).

Consider an iteration scheme

\[
\varphi_j = \xi_j + S_0 \eta_j , \quad j = 1, 2, \ldots
\]

\[
\lambda_j = \langle T\varphi_{j-1}, \varphi_0 \rangle ,
\]

where the initial terms \( \varphi_0 \) and \( \lambda_0 \) are eigenelements of \( T_0 \). It follows from (17.27) that if for some \( j = 1, 2, \ldots \),

(17.28)

\[
\xi_j^S = T \xi_j^P / \lambda_0 \quad \text{and} \quad \eta_j^S = T \eta_j^P / \lambda_0 ,
\]

then

\[
\varphi_j = \xi_j^S + S_0 \eta_j^S = T(\xi_j^P + S_0 \eta_j^P) / \lambda_0 = T \varphi_j^P / \lambda_0 ,
\]

and hence

\[
\lambda_{j+1}^S = \langle T \varphi_j^S, \varphi_0^S \rangle = \frac{1}{\lambda_0} \langle T \varphi_j^P, \varphi_0^P \rangle = \frac{1}{\lambda_0} \langle T \varphi_j^P, \varphi_0^P \rangle = \langle T \varphi_j^P, \varphi_0^P \rangle = \lambda_{j+1}^P
\]

with obvious notations. Using this result it can be proved by induction on \( j \), that for each of the iteration schemes (11.18), (11.19), and (11.35), the relations in (17.28) hold. Hence the desired result (17.25) holds for these iteration schemes. For the iteration scheme (11.31) one needs to note additionally that if \( \varphi_j^S = T \varphi_j^P / \lambda_0 \), then

\[
\mu_{j+1}^S = \langle T^2 \varphi_j^S, \varphi_0^S \rangle = \frac{1}{\lambda_0} \langle T^3 \varphi_j^P, \varphi_0^P \rangle = \frac{1}{\lambda_0} \langle T^3 \varphi_j^P, \varphi_0^P \rangle = \langle T^2 \varphi_{j-1}^P, \varphi_0^P \rangle = \mu_{j+1}^P .
\]
PROPOSITION 17.14 Under the hypotheses and notations of Proposition 17.13, the following relations hold for the Rayleigh-Schrödinger iteration scheme (11.18) and the fixed point iteration scheme (11.19):

\begin{align*}
\lambda_1^G &= \lambda_0, \\
\varphi_1^G &= \mathcal{T}\varphi_0^*/\lambda_0^* = \varphi_0^*, \\
\lambda_2^G &= \langle \mathcal{T}^2\varphi_0^*, \varphi_0^* \rangle / \lambda_0 = \lambda_1^P.
\end{align*}

Proof By definition,

\[ \lambda_1^G = \langle \mathcal{T}\varphi_0^*, \varphi_0^* \rangle = \langle \mathcal{T}_0^G\varphi_0^*, \varphi_0^* \rangle + \langle (\mathcal{T} - \mathcal{T}_0^G)\varphi_0^*, \varphi_0^* \rangle. \]

But since \( \varphi_0^* = \langle \mathcal{T}_0^G \varphi_0^* \rangle / \lambda_0 \) and

\[ \mathcal{T}_0^G(\mathcal{T} - \mathcal{T}_0^G)\varphi_0 = \mathcal{T}_0^G(\mathcal{T} - \mathcal{T}_0^G)\varphi_0 = (\mathcal{T}_0^G)^2\varphi_0 - (\mathcal{T}_0^G)^2\varphi_0 = 0, \]

we have

\[ \langle (\mathcal{T} - \mathcal{T}_0^G)\varphi_0^*, \varphi_0^* \rangle = \langle \mathcal{T}_0^G(\mathcal{T} - \mathcal{T}_0^G)\varphi_0^*, \varphi_0^* \rangle / \lambda_0 = 0. \]

Hence

\[ \lambda_1^G = \langle \mathcal{T}_0^G\varphi_0^*, \varphi_0^* \rangle = \lambda_0 \langle \varphi_0^*, \varphi_0^* \rangle = \lambda_0. \]

This proves (17.29). Next, by definition and by (17.29),

\[ \varphi_1^G = \varphi_0 + \mathcal{S}_0^G(\mathcal{T} - \mathcal{T}_0^G)\varphi_0 = \varphi_0 - \mathcal{S}_0^G(\mathcal{T}\varphi_0 - \lambda_0\varphi_0) \]

for both the iteration schemes (11.18) and (11.19). We claim that

\begin{align*}
\mathcal{S}_0^G(\mathcal{T}\varphi_0 - \lambda_0\varphi_0) &= - \frac{1}{\lambda_0} (\mathcal{T}\varphi_0 - \lambda_0\varphi_0) \\
\langle \mathcal{T}\varphi_0 - \lambda_0\varphi_0, \varphi_0^* \rangle &= \lambda_1^G - \lambda_0 = 0,
\end{align*}

and
This proves (17.32). Hence

$$\varphi_1^G = \varphi_0 + \frac{1}{\lambda_0} (T\varphi_0 - \lambda_0 \varphi_0) = \frac{T\varphi_0}{\lambda_0} = \varphi_0^S.$$ 

proving (17.30). Finally, by definition and by (17.30),

$$\lambda_2^G = \langle T\varphi_1^G, \varphi_0^* \rangle = \langle \varphi_1^G, T\varphi_0^* \rangle = \langle T\varphi_0^*, \varphi_0^* \rangle = \langle T\varphi_0^*, \varphi_0^* \rangle = \lambda_1^P.$$ //

**Remark 17.15** The relation (17.25) shows that if one knows the projection iterates $\lambda_j^P$ and $\varphi_j^P$ for one of the schemes (11.18), (11.19), (11.31) and (11.35), then the Sloan iterates $\lambda_j^S = \lambda_j^P$ and $\varphi_j^S = T\varphi_j^P/\lambda_0$ are available easily; there being no need to implement the algorithm again.

The relations (17.29), (17.30) and (17.31) provide useful checks when we perform computations with the projection, Sloan and Galerkin methods.

The relation (17.29), namely $\lambda_1^G = \lambda_0^G$, says that the generalized Rayleigh quotient $\lambda_1^G$ of $T$ at $(\varphi_0^*, \varphi_0^*)$ equals the initial eigenvalue $\lambda_0$. In other words $\lambda_1^G$ does not improve upon $\lambda_0$ as an approximation of an eigenvalue $\lambda$ of $T$. Also, the relation (17.30)

$$\varphi_1^G = T\varphi_0^*/\lambda_0 = T\varphi_0^*/\langle T\varphi_0^*, \varphi_0^* \rangle$$

tells us that the first Galerkin iterate $\varphi_1^G$ in schemes (11.18) and (11.19) is, in fact, the iterated Galerkin eigenvector proposed by Sloan in [SL], and that it coincides with the first eigenvector iterate of the power method with $\varphi_0$ and $\varphi_0^*$ as the initial terms. As such,

$$\lambda_2^G = \langle T^2\varphi_0^*, \varphi_0^* \rangle / \langle T\varphi_0^*, \varphi_0^* \rangle.$$
coincides with the second eigenvalue iterate of the power method. (See (11.36) and (11.37)).

Problems

For $x \in X$, let $T_0 x = \sum_{i=1}^{n} \langle x, x_i^* \rangle x_i$, where $x_1, \ldots, x_n$ are in $X$ and $x_1^*, \ldots, x_n^*$ are in $X^*$. Let $A = [\langle x_j^*, x_i^* \rangle]$, $1 \leq i, j \leq n$.

17.1 With the assumptions and notations of Corollary 17.2,

$$\langle x_j, \varphi_0^* \rangle = \lambda_0 y(j), \quad j = 1, \ldots, n.$$ 

17.2 Let $0 \neq \lambda_0 \in \mathbb{C}$. The algebraic and the geometric multiplicities of $\lambda_0$ as an eigenvalue of $T_0$ and of $A$ are the same. The orders of the poles at $\lambda_0$ of the resolvent operators of $T_0$ and $A$ are equal.

17.3 Let $\lambda_0$ be a nonzero semisimple eigenvalue of $A$. Let 

$\{u_1, \ldots, u_m\}$ (resp., $\{y_1, \ldots, y_m\}$) be a basis of the eigenspace of $A$ (resp., $A^*$) corresponding to $\lambda_0$ (resp., $\lambda_0^*$) such that

$$\delta_i, j y^H \lambda_0^*, \quad i, j = 1, \ldots, m.$$ 

Then analogues of Corollary 17.2 and Proposition 17.4 hold.

17.4 In Table 17.1, for $T_0^N$ we can consider

$$x_i = k(x, t_i), \quad \langle x, x_i^* \rangle = w_i x(t_i)$$

and for $T_0^F$ we can consider

$$x_i = \sum_{j=1}^{n} k(t_j, t_i) e_j, \quad \langle x, x_i^* \rangle = w_i x(t_i).$$
In both these cases, \[ \langle \xi_j, x_i^j \rangle = \langle w_j k(t_i, t_j) \rangle = A', \] say. If \( A = [w_j k(t_i, t_j)] \), then the nonzero eigenvalues of \( A \) and \( A' \) are the same, the corresponding algebraic and geometric multiplicities coincide, as do the orders of the corresponding poles of the resolvent operators of \( A \) and \( A' \). If \( \xi \) is an eigenvector of \( A \) corresponding to \( \lambda_0 \neq 0 \), then \( \xi' = [w_1 u(1), \ldots, w_n u(n)]^t \) is an eigenvector of \( A' \) corresponding to \( \lambda_0 \).

17.5 For the Rayleigh-Schrödinger scheme (11.18), the relations in (17.25) can be proved by considering the families of operators

\[
T^P(t) = T_0^P + t(T_0^P - T_0^S) \quad \text{and} \quad T^S(t) = T_0^S + t(T_0^S - T_0^P)
\]

for \( t \in \mathbb{C} \) with \( |t| \) small enough. (Hint: (10.4) and (10.5))

17.6 Let \( \lambda_0 \) be a nonzero eigenvalue of \( T_0^F \) and let \( \varphi_0 \) (resp., \( \varphi_0^* \)) be an eigenvector of \( T_0^F \) (resp., \( (T_0^F)^* \)) corresponding to \( \lambda_0 \) (resp., \( \lambda_0^* \)) such that \( \langle \varphi_0, \varphi_0^* \rangle = 1 \). Then \( \lambda_0 \) is a simple eigenvalue of \( T_0^N \), and \( \varphi_0^N = T_0^N \varphi_0 / \lambda_0 \) (resp., \( \varphi_0^N = \varphi_0^* \)) is an eigenvector of \( T_0^N \) (resp., \( (T_0^N)^* \)) corresponding to \( \lambda_0 \) (resp., \( \lambda_0^* \)) such that \( \langle \varphi_0^N, \varphi_0^* \rangle = 1 \).