1. INTRODUCTION

In this paper we discuss three cardinal numbers associated with a topological group $G$: the weight of $G$, $\omega(G)$, the local weight, $\omega_0(G)$, and $\theta(G)$, the least cardinal of a family of open sets whose intersection is a singleton. It is clear that $\theta(G) \leq \omega_0(G) \leq \omega(G)$. We give necessary and sufficient conditions for $\theta(G) = \omega_0(G) = \omega(G)$. In particular they are equal for all $\sigma$-compact locally compact Hausdorff groups.

The following notation will be used throughout the paper. If $G$ is a topological group, we denote

(a) the minimal cardinality of a family of open sets having as intersection the identity, $1$, in $G$ by $\theta(G)$;

(b) the minimal cardinality of an open basis for $G$ at $1$ by $\omega_0(G)$;

(c) the minimal cardinality of an open basis for $G$ by $\omega(G)$.

If $H$ is a topological subgroup of $G$, we write $H \leq G$.

Note that if $H \leq G$, then $\theta(H) \leq \theta(G)$, $\omega_0(H) \leq \omega_0(G)$, and $\omega(H) \leq \omega(G)$. 


PROPOSITION 1 If $G$ is any topological group then
\[ \Theta(G) \leq \omega_0(G) \leq \omega(G). \]

Proof. Clearly $\Theta(G) \leq \omega_0(G)$ and $\omega_0(G) \leq \omega(G)$. So
\[ \Theta(G) \leq \omega_0(G) \leq \omega(G). \]

We note here that if an infinite Hausdorff non-discrete topological group, $G$, satisfies the second axiom of countability, then
\[ \Theta(G) = \omega_0(G) = \omega(G) = \aleph_0. \]
Thus if $G$ is an infinite compact metrizable group, then $\Theta(G) = \omega(G) = \aleph_0$.

DEFINITION Let $U(n)$, $n \in \mathbb{N}$, be the compact group of $n \times n$ unitary matrices, and define $U = \prod_{n=1}^{\infty} U(n)$.

As $U$ is compact and metrizable $\omega(U) = \Theta(U) = \omega_0(U) = \aleph_0$.

2. COMPACT GROUPS

We use the following refinement of the Embedding Lemma, ([6], P.116) in the proof of Lemma 3. It's proof is analogous to the usual proof.

LEMMA 2 Let $\{(Y_i, T_i) \mid i \in I\}$ be a family of Hausdorff spaces, and for each $i \in I$, let $f_i$ be a mapping of a Hausdorff space $(X, \tau)$ into $(Y_i, T_i)$. Let $e : (X, \tau) \to \prod_{i \in I} (Y_i, T_i)$ be defined by $e(x) = \prod_{i \in I} f_i(x)$, for each $x \in X$. Then $e$ is a homeomorphism of $(X, \tau)$ onto the space $(e(X), \tau')$ where $\tau'$ is the subspace topology, if
(i) each \( f_i \) is continuous, and

(ii) given \( x \in X \) and any closed set \( A \) not containing \( x \), there is a finite subset \( \{i_1, i_2, \ldots, i_n\} \) of \( I \) such that

the map \( F = f_{i_1} \times f_{i_2} \times \cdots \times f_{i_n} : X \longrightarrow \prod_{j=1}^{n} (Y_j, \tau_j) \) satisfies

\[ F(x) \notin F(A). \]

**Lemma 3** Let \( G \) be a topological group and \( \{H_i \mid i \in I\} \) an infinite family of Hausdorff groups such that \( G \) is topologically isomorphic to a subgroup of the product \( \prod_{i \in I} H_i \). Then there is a subset \( J \) of \( I \), with \( \text{card } J = \omega_0(G) \), such that \( G \) is topologically isomorphic to a subgroup of \( \prod_{i \in J} H_i \).

**Proof** Without loss of generality, consider \( G \) to be a subgroup of \( \prod_{i \in I} H_i \). Let \( B = \{B_k \mid k \in K\} \) be a basis for \( G \) at the identity, \( 1 \), such that \( \text{card } K = \omega_0(G) \). For each \( k \in K \) there exists an \( O_k \) such that

\[ O_k \cap G \subseteq B_k \quad \text{where} \quad O_k = O_{k_1} \times O_{k_2} \times \cdots \times O_{k_n} \times \prod_{i \in I \setminus \{k_1, k_2, \ldots, k_n\}} H_i \]

is a member of the natural basis for \( \prod_{i \in I} H_i \) at the identity. For each \( k \in K \) put \( J_k = \{k_1, k_2, \ldots, k_n\} \) and \( J = \bigcup_{k \in K} J_k \). Then, as each \( J_k \) is finite, \( \text{card } J = \text{card } K = \omega_0(G) \).

Let \( P : \prod_{i \in I} H_i \longrightarrow \prod_{i \in J} H_i \) be the natural projection mapping. We need to show \( P : G \longrightarrow P(G) \) is a homeomorphism. As each \( p_i : G \longrightarrow H_i \) given by \( p_i(x) = p_i(\prod_{i \in I} x_i) = x_i \), is continuous, condition (i) of the Embedding Lemma is satisfied. To see condition (ii) holds, we need consider only the identity \( 1 \) and any closed set \( A \) in \( G \) such that \( 1 \notin A \). Then \( 1 \in G \setminus A \) which is open, and so there is a \( B_k \in B \) such
that \(1 \in B_k \cap G\). Therefore there is a basic open neighbourhood \(O_k\) such that \(1 \in O_k \cap G\); that is

\[1 \in (O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n} \times \prod_{i \in I \setminus \{k_1, k_2, \ldots, k_n\}} H_i) \cap G.\]

Define

\[F : G \to \prod_{j=1}^n H_{k_j} \quad \text{by} \quad F(x) = \prod_{j=1}^n P_j(x), \quad \text{for} \ x \in G.\]

Then

\[F(1) \in O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n}\]

which is open and \(F(A) \cap (O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n}) = \emptyset\)

which implies \(\overline{F(A)} \cap (O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n}) = \emptyset\). Hence \(F(1) \notin F(A)\),

and so by our Embedding Lemma, \(P\) is a homeomorphism of \(G\) onto \(P(G)\).

As \(P\) is also a homomorphism we have that \(G\) is topologically isomorphic to \(P(G)\), a subgroup of \(\prod_{i \in J} H_i\).

The countable case of the above result was used by Brooks, Morris and Saxon [2, Corollary 6].

Using a similar argument to the proof of Lemma 3, we obtain a stronger result for compact groups.

**Lemma 4** Let \(G\) be a compact group and \(\{H_i \mid i \in I\}\) an infinite family of Hausdorff groups such that \(G\) is topologically isomorphic to a subgroup of the product \(\prod_{i \in I} H_i\). Then there is a subset \(J\) of \(I\), with \(\text{card } J = \theta(G)\), such that \(G\) is topologically isomorphic to a subgroup of \(\prod_{i \in J} H_i\).

**Proof** Again, consider \(G\) to be a subgroup of \(\prod_{i \in I} H_i\), and let

\[\Phi(G) = \{U_k \mid k \in K\}\]

be a family of open sets of \(G\) such that

\[\text{card } \Phi(G) = \theta(G) \quad \text{and} \quad \bigcap_{k \in K} U_k = \{1\}.\]

For each \(k \in K\) there is an
open set \( O_k \) such that \( O_k \cap G \subseteq U_k \) where

\[
O_k = O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n} \times \prod_{i \in I \setminus \{k_1, k_2, \ldots, k_n\}} H_i
\]

is a member of the natural basis for \( \prod H_i \) at the identity. For each \( k \in K \) put \( J_k = \{k_1, k_2, \ldots, k_n\} \) and \( J = \bigcup_{k \in K} J_k \). Then \( \text{card } J = \text{card } K = \theta(G) \).

Let \( P : \prod H_i \to \prod H_i \) be the natural projection mapping. Then \( P : G \to \prod H_i \) is a continuous injective homomorphism. As \( G \) is compact, \( G \) is topologically to \( P(G) \), from which the result follows. //

The next lemma is an immediate consequence of the Peter-Weyl Theorem ([17], P.62).

**LEMMA 5** If \( G \) is a compact Hausdorff group, then it is topologically isomorphic to a subgroup of a product of copies of the group \( \mathbb{U} \).

**THEOREM 1** [3, 28.58] Let \( G \) be an infinite compact Hausdorff group. Then \( \theta(G) = \omega_0(G) = \omega(G) \).

**Proof** By Lemma 5, we can, without loss of generality, assume that \( G \) is a subgroup of \( \mathbb{U}^{\text{card } I} \), for some index set \( I \). But using Lemma 4 we have that \( G \) is topologically isomorphic to a subgroup of \( \mathbb{U}^{\theta(G)} \).

So \( \omega(G) \leq \omega(\mathbb{U}^{\theta(G)}) \)

\[
= \max \{\omega(\mathbb{U}), \theta(G)\}
\]

\[
= \max \{\theta(G), \theta(G)\}
\]

\[
= \theta(G), \text{ as } \theta(G) \text{ is infinite.}
\]
But $\theta(G) \leq \omega(G)$ from Proposition 1. Thus $\theta(G) = \omega(G)$, from which it follows that $\omega_0(G) = \theta(G) = \omega(G)$. //

Hulanicki [3] proved that $\text{card } G = 2^{\theta(G)}$ for $G$, any infinite compact Hausdorff group, or any infinite connected locally compact Hausdorff group. Elsewhere we shall give quite a different proof of a more general result. Here we point out a corollary to this result and Theorem 1.

**THEOREM 2** [3, 28.58] Let $G$ be any infinite compact Hausdorff group. Then $\text{card } G = 2^{\theta(G)} = 2^{\omega_0(G)} = 2^{\omega(G)}$.

3. ALMOST CONNECTED GROUPS

**DEFINITION** A locally compact Hausdorff group is said to be *almost connected* if the group $G/G_0$ is compact, where $G_0$ is the connected component of the identity. (See [1].)

Of course, the class of almost connected groups includes the class of compact Hausdorff groups and the class of connected locally compact Hausdorff groups.

**THEOREM 3** Let $G$ be any infinite almost connected group. Then $\theta(G) = \omega_0(G) = \omega(G)$ and $\text{card } G = 2^{\theta(G)} = 2^{\omega_0(G)} = 2^{\omega(G)}$.

**Proof** By Mostert ([7], Theorem 8) $G$ is homeomorphic to $G_0 \times G/G_0$.

The Iwasawa Structure Theorem ([6], p.118) says that the connected locally compact Hausdorff group $G_0$ is homeomorphic to $\mathbb{R}^n \times K$. 
where $K$ is a compact group, $\mathbb{R}$ is the topological group of real numbers with the usual topology, and $n$ is a non-negative integer. As $G/G_0$ is compact, we have that $G$ is homeomorphic to $\mathbb{R}^n \times K'$ where $K'$ is the compact Hausdorff group $K \times G/G_0$.

If $K'$ is finite, then clearly $\theta(G) = \omega_0(G) = \omega(G) = \aleph_0$, and $\text{card } G = 2^{\aleph_0}$.

If $K'$ is infinite, then $\theta(G) = \theta(\mathbb{R}^n \times K') = \theta(\mathbb{R}^n) \times \theta(K')$.

Since $\theta(\mathbb{R}^n) = \aleph_0$ we have that $\theta(G) = \theta(K')$. Similarly, $\omega_0(G) = \omega_0(K')$ and $\omega(G) = \omega(K')$. Then, by Theorem 1, we have $\theta(G) = \omega_0(G) = \omega(G)$.

Further, $\text{card } G = \text{card } \mathbb{R}^n \times \text{card } K'$

$$= 2^{\aleph_0} \times 2^{\theta(K')}$$

$$= 2^{\aleph_0} + \theta(K')$$

$$= 2^{\theta(K')}.$$

Hence, $\text{card } G = 2^{\theta(G)} = 2^{\omega_0(G)} = 2^{\omega(G)}$.

4. THE GENERAL CASE

For $G$, any topological group, we denote the least cardinality of a family of compact sets whose union is $G$ by $\gamma(G)$.

**Lemma 6** Every locally compact Hausdorff group has an open almost connected subgroup.
Proof. Let $G$ be any locally compact Hausdorff group and let $G_0$ be the component of the identity. Let $f : G \to G/G_0$ be the quotient mapping. Then the quotient group $G/G_0$ is a locally compact totally disconnected group and so has a basis of compact open subgroups, ([7], p.21). Take one such compact open subgroup, $K$. Then $f^{-1}(K) = H$ is an open subgroup of $G$. As $H$ is open and therefore closed, $G_0 \subseteq H$, and so $H_0 = G_0$. This implies $H/H_0 = H/G_0 = K$. Hence $H$ is a locally compact Hausdorff group, and $H/H_0$ is compact, from which the result follows. 

THEOREM 4 Let $G$ be any infinite locally compact Hausdorff group.

Then (i) $\omega_0(G) = \theta(G)$; (ii) $\omega(G) = \max\{\omega_0(G), \gamma(G)\}$ and (iii) $\text{card } G = \max\{2^{\omega_0(G)}, \gamma(G)\}$.

Proof (i) Let $H$ be an open almost connected subgroup of $G$. Then $\omega_0(H) = \theta(H)$ by Theorem 3. We show that $\omega_0(G) = \omega_0(H)$ and $\theta(G) = \theta(H)$, from which the result will follow.

Let $B_0$ be a basis for $H$ at the identity with $\text{card } B_0 = \omega_0(H)$. Then $B_0$ is also a basis for $G$ at the identity. So $\omega_0(G) \leq \omega_0(H)$, and hence $\omega_0(G) = \omega_0(H)$.

Let $\phi(H)$ be a family of open sets in $H$ whose intersection is the identity. Then $\phi(H)$ is also a family of open sets in $G$ whose intersection is the identity, as $H$ is open. So $\theta(G) \leq \theta(H)$, and hence $\theta(G) = \theta(H)$.
(ii) If $G$ is compact $\omega(G) = \omega_0(G)$ from Theorem 3, and
\[ \gamma(G) = 1, \] which implies $\omega(G) = \max\{\omega_0(G), \gamma(G)\}$. So assume $G$ is non-compact. Let $\{g_i \mid i \in I\}$ be a complete set of coset representatives of $H$ in $G$, and let $\text{card } I = m$. We show firstly that
\[ \omega(G) = \max\{\omega(H), m\}. \]
Let $B$ be a basis for $H$. It is clear that
\[ \{g_i B \mid B \in B, i \in I\} \] is a basis for $G$ as $H$ is open. Thus
\[ \omega(G) \leq \max\{\omega(H), m\}. \]
We know that $\omega(H) \leq \omega(G)$, and, as each coset is open and must contain a basic open set of $G$, $\omega(G) \geq m$. Hence
\[ \omega(G) = \max\{\omega(H), m\}. \]

As $H$ is almost connected, it is homeomorphic to $\mathbb{R}^n \times K$, where $K$ is a compact group and $n \in \mathbb{N}$. Therefore $\gamma(H) \leq \aleph_0$.

Let $\{A_n \mid n \in \mathbb{N}\}$ be a family of compact sets whose union is $H$. Then $\{g_i A_n \mid i \in I, n \in \mathbb{N}\}$ is a family of compact sets whose union is $G$, and therefore $\gamma(G) \leq \max\{\aleph_0, m\}$. Let $\{K_j \mid j \in J\}$ be a family of compact sets whose union is $G$ and with $\text{card } J = \gamma(G)$.

Then each $K_j$, being compact, is contained in the union of a finite number of cosets; that is, $K_j \subseteq \bigcup_{k=1}^{m_j} g_{i_k} H$ for $m_j \in \mathbb{N}$. So
\[ \gamma(G) = \text{card } J \geq m. \]
Now, clearly, $\gamma(G) \geq \aleph_0$, and so we get
\[ \gamma(G) = \max\{\aleph_0, m\}. \]

Finally, we have $\omega(G) = \max\{\omega(H), m\}$
\[ = \max\{\omega_0(G), m\}, \text{ as } \omega(H) = \omega_0(H) = \omega_0(G) \]
\[ = \max\{\omega_0(G), m, \aleph_0\}, \text{ as } \omega(G) \text{ is infinite} \]
\[ = \max\{\omega_0(G), \gamma(G)\}. \]
(iii) If $G$ is compact we already have that
\[
\text{card } G = 2^{\omega_0(G)} = \max \{2^{\omega_0(G)}, \gamma(G)\}
\]
from Theorem 2, so again assume $G$ is non-compact. Then
\[
\text{card } G = \text{card } H \cdot m = \max \{2^{\omega_0(H)}, m\} = \max \{2^{\omega_0(G)}, \gamma(G)\}.
\]

We note that Hulanicki's Fundamental lemma is a corollary to the above theorem.

**COROLLARY 1** ([4], p.67) If $G$ is an infinite locally compact Hausdorff group, then $\text{card } G \geq 2^{\theta(G)}$.

**COROLLARY 2** Let $G$ be an infinite locally compact Hausdorff group. Then the following are equivalent

(i) $\omega(G) = \omega_0(G)$;
(ii) $\gamma(G) \leq \omega_0(G)$.

**COROLLARY 3** ([3], p.100) If $G$ is an infinite $\sigma$-compact locally compact Hausdorff group, then $\omega(G) = \omega_0(G) = \theta(G)$.

**COROLLARY 4** ([4], p.69) If the locally compact Hausdorff group, $G$, is $2^{\theta(G)}$-compact, then $\text{card } G = 2^{\theta(G)}$.

**REFERENCES**


Department of Mathematics
La Trobe University
Bundoora Vic. 3083
Australia