Remarks on Non-Commutative Banach Function Spaces

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The purpose of this note is to outline an approach to the duality theory of non-commutative Banach function spaces which extends earlier work of Yeadon [Y1],[Y2]. The details will appear elsewhere.

Let \( \mathcal{M} \) be a semifinite von Neumann algebra with a semifinite normal trace \( \tau \) and let \( \hat{\mathcal{M}} \) be the *-algebra of \( \tau \)-measurable operators (in the sense of Nelson [N]) affiliated with \( \mathcal{M} \). For each \( x \in \mathcal{M} \) and \( 0 < t \in \mathbb{R} \), the generalized singular value \( \mu_t(x) \) is defined to be

\[
\mu_t(x) = \inf \{ \lambda \geq 0 : \tau(1 - e_\lambda) \leq t \}
\]

where \( \{ e_\lambda \} \) denotes the spectral resolution of \( |x| \). Our approach is based on the following result.

**Proposition 1.** If \( x, y \in \hat{\mathcal{M}} \), then

\[
\sup \left\{ \int_E |\mu_t(x) - \mu_t(y)| \, dt : |E| \leq u \right\} \leq \int_0^u \mu_t(x - y) \, dt
\]

for each \( u \geq 0 \).

The preceding result is a common generalization of the well known inequality of Markus ([M], Theorem 5.4) for compact operators and that of Lorentz and Shimogaki [LS] for the case that \( \mathcal{M} \) is abelian. A similar inequality has been established by Hiai and Nakamura [HN] via the real interpolation method. Our present approach however is direct and is not based on interpolation methods.

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Suppose now that $L_\rho \subseteq L^0(\mathcal{M}, dm)$ is a rearrangement invariant Banach function space for which $\rho$ is an invariant Fatou norm (see, for example [KPS], Chapter II). The non-commutative space $L_\rho(\mathcal{M})$ is defined by setting

$$L_\rho(\mathcal{M}) = \{ x \in \mathcal{M} : \mu(x) \in L_\rho \}$$

and for $x \in L_\rho(\mathcal{M})$, $\| x \|_\rho$ is defined to be $\rho(\mu(x))$. The generalized Markus inequality given by Proposition 1 may be used to show that the spaces $L_\rho(\mathcal{M})$ are Banach spaces. We define the space

$$L_\rho(\mathcal{M})^\times = \{ x \in \mathcal{M} : xy \in L^1(\mathcal{M}) \text{ for all } y \in L_\rho(\mathcal{M}) \}.$$ 

The space $L_\rho(\mathcal{M})^\times$ may be identified with a subspace of the Banach dual $L_\rho(\mathcal{M})^*$. If $L^\times_\rho$ denotes the (Köthe) associate space of $L_\rho$ and if $L_\rho(\mathcal{M})^\times$ is equipped with the norm induced by $L_\rho(\mathcal{M})^*$, then we have the following identification.

**Proposition 2.**

$$L_\rho(\mathcal{M})^\times = L^\times_\rho(\mathcal{M})$$

In turn, the non-commutative associate space $L^\times_\rho(\mathcal{M})$ may be identified via a Radon-Nikodym type theorem as that subspace of the Banach dual $L_\rho(\mathcal{M})^*$ consisting of normal linear functionals.

**References**


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