SINGULAR INTEGRALS ON BMO

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Let \( f \) be a locally integrable function on \( \mathbb{R}^n \). We say \( f \) has bounded mean oscillation, \( f \in \text{BMO} \), if

\[
\sup_B \inf_{c \in \mathbb{R}} \frac{1}{|B|} \int_B |f(y) - c| \, dy < +\infty,
\]

where the supremum is taken over all balls \( B \subset \mathbb{R}^n \). Identifying functions which differ by an additive constant a.e. makes BMO a Banach space with norm \( \| \cdot \|_{\text{BMO}} \) equal to the left hand side of (1). Note that \( L^\infty \) is a proper subset of BMO, since \( \log|x| \in \text{BMO} \).

Let \( K \) be a locally integrable function on \( \mathbb{R}^n \setminus \{0\} \) such that

\[
Tf(x) = \lim_{\epsilon \downarrow 0} \int_{\{ |y| > \epsilon \}} K(y)f(x-y) \, dy
\]

is a bounded operator on \( L^2 \). We say \( K \) satisfies condition \( H_r \), \( 1 \leq r < \infty \), if there is a non-decreasing function \( s \) on \( (0,1) \) such that

\[
\sum_{j=1}^{\infty} s(2^{-j}) < +\infty \quad \text{and} \quad \left[ \int_{\{x: R < |x| < 2R\}} |K(x-y) - K(x)|^r \, dx \right]^{1/r} \leq s\left(\frac{1}{2}\right) R^{-n/r'}, \quad \text{for } |y| < R/2.
\]

Define \( H_\infty \) by the obvious modification.

If \( f \in L^\infty \) is supported on a set of finite measure and \( K \in H_1 \), then \( Tf \) exists a.e. (i.e., the limit exists and is finite), \( Tf \in \text{BMO} \), and \( \|Tf\|_{\text{BMO}} \leq C\|f\|_{\text{BMO}} \) [2]. On the other hand, if \( f \) is merely bounded, then without a suitable modification \( Tf \) may fail to exist on a set of positive measure. For example, if \( f(x) = x_i^E(x) \) is the characteristic function of \( E = \{x \in \mathbb{R}^n: x_i > 0, i=1,...,n\} \), then the Riesz transforms of \( f \), defined by the kernels \( K_j(x) = \frac{x_j}{|x|^{n+1}}, j=1,...,n \), are infinite a.e.
Let $I(x)$ be a constant function on $\mathbb{R}^n$. We say $K \in H^+_r$, $1 \leq r \leq \infty$, if $K \in H_r$, $TI = 0$, and $\sum_{j=1}^{\infty} s(2^{-j}) < +\infty$.

**THEOREM:** Suppose $K \in H^+_r$, $1 < r \leq \infty$, and $f \in \text{BMO}$. Either $Tf$ fails to exist almost everywhere or $Tf \in \text{BMO}$ and

$$\|Tf\|_{\text{BMO}} \leq C\|f\|_{\text{BMO}}.$$

The constant $C$ is independent of $f$.

Given $x \in \mathbb{R}^n$ and $\delta > 0$, set $B(x, \delta) = \{y \in \mathbb{R}^n : |x - y| \leq \delta \}$. For $B = B(x, \delta)$, let $f_B = \frac{1}{|B|} \int_B f(y)\mathrm{d}y$. The proof of the theorem is based on the following lemma. (See [4].)

**LEMMA:** Let $1 \leq p < \infty$. There is a constant $C$ depending on $n$ and $p$ so that for $f \in \text{BMO}$, $B = B(x, \delta)$, and $k \geq 1$,

$$\left( \int_{B(x, 2^k \delta)} |f(y) - f_B|^p \mathrm{d}y \right)^{1/p} \leq Ck(2^k \delta)^n/p\|f\|_{\text{BMO}}.$$

We now sketch a proof of the theorem. (See [6,4].) Suppose $E = \{x \in \mathbb{R}^n : Tf(x) \text{ exists} \}$ has positive measure. Let $x_0$ be a point of density of $E$ and $\delta > 0$. Set $B = B(x_0, \delta)$ and $\bar{B} = B(x_0, 4\delta)$. Write $f(x) = f_B + \left[ f(x) - f_B \right] \chi_B(x) + \left[ f(x) - f_B \right] \chi_{\mathbb{R}^n \setminus B}(x) = f_B + g_B(x) + h_B(x)$. Since $f_B$ is constant, $Tf_B = 0$. By the lemma, $g_B \in L^2$ and

$$\int_{\bar{B}} |Tg_B(y)|\mathrm{d}y \leq |\bar{B}|^{1/2} \|Tg_B\|_2 \leq C_1 |\bar{B}|^{1/2} \|g_B\|_2 \leq C_2 |\bar{B}| \|f\|_{\text{BMO}}.$$

It follows that $Tg_B$ exists a.e. so that $Tf$ exists at almost every point such that
Since $x_0$ is a point of density of $E$ and $Tg_B$ exists a.e., there is a point $y_0 \in B(x_0,\delta)$ such that $Th_B(y_0) = Tf(y_0) - Tg_B(y_0)$ exists. Suppose $x \in B$. Set $A_j = \{ z \in \mathbb{R}^n : 2^j \delta < |x_0 - z| \leq 2^{j+1} \delta \}$. By the lemma, since $K \in H^+_\tau$ and $|x-y_0| \leq 2\delta$,

$$\|T_B(x) - Th_B(y_0)\| \leq \int |K(x-z) - K(y_0-z)| |h_B(z)| \, dz$$

$$= \sum_{j=2}^{\infty} \int_{A_j} |K(x-z) - K(y_0-z)| |f(z) - f_B| \, dz$$

$$\leq \sum_{j=2}^{\infty} \left( \int_{A_j} |K(x-z) - K(y_0-z)|^{1/r} \, dz \right)^{1/r'} \left( \int_{B(x_0,2j+1\delta)} |f(z) - f_B|^{r'} \, dz \right)^{1/r'}$$

$$\leq C \sum_{j=2}^{\infty} s \left( \frac{|x-y_0|}{2^j \delta} \right)^{2j+1} j^{(2j+1)\delta/n} \|f\|_{\text{BMO}}$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} j\|f\|_{\text{BMO}} = C'\|f\|_{\text{BMO}}.$$  

As a consequence of (3), $Th_B$ exists a.e. in $B$, which implies $Tf$ exists a.e. in $B$. By considering only $B(x_0,\delta)$ with $\delta$ a positive integer, it follows that $Tf$ exists a.e. in $\mathbb{R}^n$.

To show $\|Tf\|_{\text{BMO}} \leq C\|f\|_{\text{BMO}}$, fix $B = B(x,\delta)$ and choose $y_0$ as before. By (2) and (3),

$$\frac{1}{|B|} \int_B |Tf(y) - Th_B(y_0)| \, dy \leq \frac{1}{|B|} \int_B |Tg_B(y)| \, dy + \frac{1}{|B|} \int_B |Th_B(y) - Th_B(y_0)| \, dy$$

$$\leq C \|f\|_{\text{BMO}}.$$  

Since $B$ was arbitrary, we see that $Tf \in \text{BMO}$ and $\|Tf\|_{\text{BMO}} \leq C\|f\|_{\text{BMO}}$. 

Let $\sum_{n=1}^{\infty} = \{x \in \mathbb{R}^n : |x| = 1\}$ and $\rho$ be a rotation of $\sum_{n=1}^{\infty}$ with
$$|\rho| = \sup_{x \in \sum_{n=1}^{\infty}} |x-\rho x|.$$ Suppose $K(x) = \frac{\Omega(x)}{|x|^n}$, where $\Omega$ is homogeneous of degree 0 and
$$\int_{\sum_{n=1}^{\infty}} \Omega(x) d\sigma(x) = 0.$$ Let $\omega_\tau$ be the $L^\tau$ modulus of continuity of $\Omega$ on $\sum_{n=1}^{\infty}$, $\omega_\tau(x) = \sup_{|y-x| \leq \delta} \frac{|\Omega(x)-\Omega(y)|^{1/\tau}}{\delta^{1/\tau}}$. (For $\tau = \infty$, use the $L^\infty$ norm). Then $K \in H_{-\infty}^\tau$ if
$$\int_0^1 \omega_\tau(\delta) d\delta < + \infty.$$ (This is a slightly stronger condition than the $L^\tau$-Dini condition, which implies $K \in H_{-\infty}^\tau$.) In particular, if $\Omega \in \text{Lip}(\alpha)$, $\alpha > 0$,
$$|\Omega(x)-\Omega(y)| \leq C |x-y|^\alpha,$$
then $\Omega \in H_{-\infty}^\tau$. Thus, the Riesz transforms satisfy the theorem.

REFERENCES

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