ABSTRACT. Let $X$ be a Banach space and $C$ be a closed, convex subset of $X$ such that $N(C)$, the subset of its non support points, is non empty. We investigate differentiability properties of convex functions defined on $N(C)$ and recover many results known to be true in the case $N(C) = \text{int}(C)$.

0. Notation. Let $X$ be a Banach space, $X^*$ be its topological dual and $C \subset X$ be a closed convex set. We shall denote by $S(C)$ the set of all support points of $C$ and by $N(C)$ the set of all non-support points of $C$. For $x \in C$ let $C_x = \{y \in X; x+ty \in C \text{ for some } t>0\}$ be the cone generated by $C$ from $x$.

Recall that $N(C)$, if non-empty, is a convex, dense $G_δ$ subset of $C$, in fact a Baire space. If $C$ has interior points, then $N(C)$ is exactly the interior of $C$. Also $x \in N(C)$ iff $\text{cl}(C_x) = X$.

1. THEOREM. Let $C$ be a closed, convex set of $X$ with $N(C) \neq \emptyset$ and $A$ be a relatively open subset of $N(C)$. Let $f : N(C) \to \mathbb{R}$ be convex and such that $f|A$ is locally Lipschitz. Then

(i) $\partial f(x) \neq \emptyset$ for all $x \in A$;

(ii) $\partial f(x)$ is a weak $^*$ compact subset of $X^*$;

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(iii) the subdifferential map $\partial f: A \to (2^{X^*}, \text{weak}^*)$ is usco and locally bounded.

**Proof.** The first two assertions are slightly more general than the corresponding assertions in Theorem 1 of [V]; the proof given there is also valid in the actual context. The third assertion was noticed in [R1].

**Remark.** As a matter of fact, the first assertion is true for $A$ relatively open in $C$ and $f: C \to \mathbb{R}$ convex and locally Lipschitz on $A$ (see [N]). The proof given in [V] can be used to obtain this result too.

Conversely, assume that $C$ is a convex subset of $X$, $f: C \to \mathbb{R}$ is convex, $\partial f(x) \neq \emptyset$ for all $x \in C$ and $\partial f: C \to 2^{X^*}$ is locally bounded. Then it is easily seen that $f$ is locally Lipschitz on $C$. As a matter of fact, as noticed by D. Noll [N], the following result holds true: if $f: C \to \mathbb{R}$ is convex and $\partial f(x) \neq \emptyset$ for $x$ in a Baire space which is dense in $C$, then $f$ is locally Lipschitz at the points of a dense, relatively open subset of $C$. A proof of a slightly more general result can also be found in [R2]. In what follows we shall present a different, more direct proof of the same result.

2. **PROPOSITION.** Let $C$ be a convex subset of a Banach space $X$ and $f: C \to \mathbb{R}$ be convex. Let $A$ be a Baire space contained in $C$ such that $\partial f(x) \neq \emptyset$ for $x \in A$. Then there exists a dense in $A$ set $D$ such that the restriction of $f$ to $A$ is locally Lipschitz at each point of $D$. If in addition $f$ is lsc on $\text{cl}_C(A)$, the closure of $A$ in $C$, then the restriction of $f$ to $\text{cl}_C(A)$ is locally Lipschitz at each point of $D$. 
Proof. Notice first that $f$ is lsc on $A$ (since $\exists f(x) \neq \emptyset$ for $x \in A$). For each $n \geq 1$ let

$$F_n = \{ x \in A; \exists f(x) \cap nB^* \neq \emptyset \}$$

($B^*$ is the closed unit ball in $X^*$). Clearly $A = \bigcup F_n$.

STEP I. $F_n$ is closed in $A$. Indeed let $(x_k)$ be a sequence in $F_n$ converging to $x \in A$. For each $k$ choose $h_k \in \partial f(x_k) \cap nB$. Since $nB$ is bw compact, there exists $h \in nB^*$, a bw cluster point of the sequence $(h_k)$. Let $y \in C$ and $\varepsilon > 0$; then there exists $k$ such that $|h_k(y) - h(y)| \leq \varepsilon$, $|h_k(x_i) - h(x_i)| \leq \varepsilon$, $i \geq 1$, $|h_k(x) - h(x)| \leq \varepsilon$, $\|x_k - x\| < \varepsilon / \|h\|$, $f(x_k) > f(x) - \varepsilon$. We have:

$$h(y-x) = h_k(y-x_k) + h(y) - h_k(y) + h_k(x_k) - h(x_k) + h(x_k) - h(x) \leq f(y) - f(x_k) + \varepsilon + \varepsilon + \|h\| \|x_k - x\| \leq f(y) - f(x) + 4\varepsilon.$$  

Since $\varepsilon$ is arbitrary, $h(y-x) \leq f(y) - f(x)$, showing that $h$ is a subgradient of $f$ at $x$. By construction $h \in nB^*$, so $h \in F_n$.

STEP II. $f$ is Lipschitz on $F_n$ with Lipschitz constant $n$. Let $x, y \in F_n$. For $h \in \partial f(x) \cap nB^*$, we obtain $h(y-x) \leq f(y) - f(x)$. So

$$f(x) - f(y) \leq h(x-y) \leq n\|x-y\|.$$  

By symmetry, $f(y) - f(x) \leq n\|x-y\|$, proving the assertion.

STEP III. Construction of $D$. Let $G_n$ be the interior of $F_n$ in $A$ and let $D = \bigcup G_n$. Since $A$ is Baire, $D$ is dense in $A$. The first assertion is proved.

Assume now that $f$ is lsc on $\overline{C}(A)$. 
STEP IV. \( f|_{cl_C(A)} \) is locally Lipschitz at each point of \( D \). Let \( z \in D \). There exists \( \delta > 0 \) such that \( B(z, \delta) \cap A \subset F_n \) for some \( n \). Let \( x, y \in B(z, \delta) \cap cl_C(A) \) and let \( \varepsilon > 0 \). Since \( f \) is lsc at \( x \) there exists \( x' \in B(z, \delta) \cap B(x, \varepsilon) \cap A \) such that
\[
 f(x') > f(x) - \varepsilon.
\]
Next pick \( y' \in B(z, \delta) \cap B(y, \varepsilon) \cap A \) and \( h \in \partial f(y') \). We have
\[
 h(y - y') \leq f(y) - f(y').
\]
Combining the last two inequalities we get
\[
 f(x) - f(y) = f(x) - f(x') + f(x') - f(y') + f(y') - f(y) \leq \varepsilon + n \|x' - y'\| + h(y' - y) \\
 \leq \varepsilon + n(2\varepsilon + \|x - y\|) + n\varepsilon.
\]
Since \( \varepsilon \) is arbitrary, \( f(x) - f(y) \leq n \|x - y\| \), which proves the last assertion.

It is natural now to investigate the Gâteaux and Fréchet differentiability of such functions. A first result in this direction was obtained in [V] where, under the assumption that \( X \) is separable, a generalization of Mazur's theorem was given. That result was extended in [R1] to a larger class of Banach spaces. In what follows we shall extend some results of Stegall [S1, S2] to our context and then use them to reobtain the results in [R1].

Let \( B \) be a subset of a Banach space \( Y \). Let \( T_x(B) \subset Y \) consist of those \( v \in Y \) with the following property: there exists a sequence \( (t_n) \) of positive real numbers, decreasing to 0 and such that \( x + t_nv \in B \) for each \( n \). If \( B \) is convex, \( T_x(B) = B_x \).
DEFINITIONS. Let $X$, $Y$ be Banach spaces, $B \subseteq Y$.

1. A function $h: B \to X$ is called Gateaux differentiable at $x \in B$ if there exists $h_x: Y \to X$ linear and continuous such that

$$h_x(v) = \lim_{t \to 0} (h(x+tv)-h(x))/t,$$

for all $v \in T_x(B)$.

2. A function $h: B \to X$ is called Fréchet differentiable at $x \in B$ if there exists $h_x: Y \to X$ linear and continuous such that the function $0_{h,x}: B \to X$ defined by

$$0_{h,x}(y) = \begin{cases} (h(y)-h(x)-h_x(y-x))/\|y-x\| & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}$$

is $(\|\cdot\|, \|\cdot\|)$ continuous at $x$.

Observe that in the above definitions the linear continuous map $h_x$ is in general not unique. However if $\text{cl aff } T_x(B) = Y$, (for example when $B$ is convex and $N(B) \neq \emptyset$) then $h_x$ is unique. If $B$ is open we recover the usual definitions.

3. THEOREM. Let $X$, $Y$ be Banach spaces, where $X$ is Asplund (resp. $X \subseteq$ class S (see [S1, S2])). Let $B \subseteq Y$ be a Baire space, $C \subseteq X$ be a closed convex set with $N(C) \neq \emptyset$ and $U$ be a relatively open subset of $N(C)$. Let $h: B \to U$ be continuous on $B$ and Fréchet (resp. Gateaux) differentiable on a dense $G_δ$ subset of $B$ and $f: N(C) \to R$ be convex and locally Lipschitz on $U$. Then $f \circ h$ is Fréchet (resp. Gateaux) differentiable on a dense $G_δ$ subset of $B$.

Proof. We shall prove the assertion about Fréchet differentiability. The other one can be proved similarly. By Theorem 1 it follows that $f: U \to (2^{X^*}, \text{weak}^*)$ is usco and locally bounded. Then,
the set valued map $G: B \to (2^X, \text{weak}^*)$ defined by $G(x) = \mathcal{E}f(h(x))$ is usco and locally bounded. Since $X$ is Asplund, it follows from Lemma 6.12 and Proposition 6.3 (b) in [P] that there exist a selection $\sigma : B + X^*$ for $G$ and a dense $G_0$ subset $D_1$ of $B$ such that $\sigma$ is $(\| \cdot \|, \| \cdot \|)$ continuous at each point of $D_1$. Let $D_2$ be the dense $G_0$ subset of $B$ on which $h$ is Fréchet differentiable. Then $D = D_1 \cap D_2$ is a dense $G_0$ subset of $B$. For $x \in D$ define $F_x : Y \to R$ by $F_x = \sigma(x) \circ h_x$, ($h_x$ is the Fréchet differential of $h$ at $x$). Clearly $F_x$ is linear and continuous. Let $x, y \in B$; then $\sigma(x) \in \mathcal{E}f(h(x))$, $\sigma(y) \in \mathcal{E}f(h(y))$ and

$$0 \leq f \circ h(y) - f \circ h(x) - \sigma(x)(h(y) - h(x)) \leq (\sigma(y) - \sigma(x))(h(y) - h(x)).$$

Using the Fréchet differentiability of $h$ at $x$, we get

$$0 \leq f \circ h(y) - f \circ h(x) - \sigma(x)(h_x(y-x) + \|y-x\| O_{h,x}(y))$$

or

$$\leq (\sigma(y) - \sigma(x))(h_x(y-x) + \|y-x\| \cdot O_{h,x}(y))$$

$$\leq \sigma(y)(O_{h,x}(y)) + ((\sigma(y) - \sigma(x))(h_x(y-x))) / \|y-x\|$$

$$\leq \sigma(y)(O_{h,x}(y)) + \|\sigma(y) - \sigma(x)\| \cdot h_x \|$$

$$\leq \sigma(y)(O_{h,x}(y)) + \|\sigma(y) - \sigma(x)\| \cdot h_x \|.$$
hood of \( x \) and \( \partial_h x \) is continuous at \( x \), \( 0_{f \circ h} x \) is continuous at \( x \) which proves the theorem.

**Note.** Stegall ([4], [5]) proved the above results in the case when \( B \) is open and \( U = X \).

Taking \( B = U \) and \( h = Id \) we obtain the following corollary.

**4. COROLLARY [R1].** Let \( X \) be a Banach space of class \( S \) (resp. Asplund), \( C \subseteq X \) be closed and convex with non-empty \( N(C) \) and \( f: N(C) \to \mathbb{R} \) be convex and locally Lipschitz on a dense, relatively open subset of \( N(C) \). Then \( f \) is Gâteaux (resp. Fréchet) differentiable on a dense \( G_δ \) subset of \( N(C) \).

**Note.** In view of Proposition 3, in the preceding Corollary one can replace the locally Lipschitz assumption by: "\( f \) is lsc on \( N(C) \) and \( \partial f(x) \neq \emptyset \) for all \( x \) in a Baire, dense subset of \( N(C) \)". The Fréchet differentiability part in the above corollary was also proved in [N] for convex, locally Lipschitz functions defined on Baire convex sets.

Another result that is true in this context is Kenderov's theorem: In a Banach space \( X \), for every convex locally Lipschitz function \( f \) on \( N(C) \) there exists a dense \( G_δ \) subset of \( N(C) \) at each point \( x \) of which \( \partial f(x) \) lies in a face of a sphere of \( X^* \). This can be proved as in [G, p.135]. Using this, one can proceed as in [G, Theorem 15, p.137] and obtain that the Gâteaux differentiability assertion in Corollary 4 is true if \( X \) can be equivalently renor-
med such that $X^*$ is rotund.

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References


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