This lecture contains work done jointly with P. Vrbová [6] and will develop some variations on a theme which goes back to Johnson and Sinclair [2].

The general question under scrutiny is that of continuity of intertwiners: if \( S, T \) are given linear operators on the vector spaces \( X, Y \), respectively, then we consider the space

\[
\mathcal{I}(S, T) := \{ \theta : X \to Y \mid \theta \text{ linear}, \ C(S, T)^n \theta = 0, \ \text{some } n \in \mathbb{N} \},
\]

where \( C(S, T)^n \) is the \( n \)th composition of the map

\[
C(S, T) : \theta \to S\theta - \theta T,
\]

and try to decide when \( \mathcal{I}(S, T) \) consists of continuous maps (provided \( X, Y \) are Banach spaces and \( S, T \) are continuous).

The interest in the space \( \mathcal{I}(S, T) \) stems from the fact that it contains many significant classes of maps:

If \( A, B \) are Banach algebras and \( \theta : A \to B \) is an algebra homomorphism then \( \theta \in \mathcal{I}(\theta(a), a) \) for any \( a \in A \) in the sense that

\[
\theta(a)\theta(x) - \theta(ax) = 0
\]

for all \( x \in A \).

Another class of examples emerges if \( X \) is a Banach algebra and \( Y \) is a commutative Banach \( X \)-module; if \( D : X \to Y \) is a module derivation then \( C(a, a)^2 D = 0 \), as an easy calculation will show.

To state the main results we will need a few facts about the algebraic spectral subspaces \( E_S(A) \) of a linear operator \( S \) on a vector space \( Y \): given a subset \( A \subseteq C \),
$E_S(A)$ is the maximal subspace of $Y$ among all subspaces $Z$ for which

$$(S - \lambda)Z = Z \quad \text{for all } \lambda \notin A.$$  

In particular $E_S(\emptyset)$ is the largest $S$-divisible subspace of $Y$. It is clear that

$$E_S(A) \subseteq \cap_{\lambda \notin A, n \in \mathbb{N}} (S - \lambda)^n Y$$

and in some cases, e.g. when $S$ is a generalized scalar operator, we have equality [5]. The only instance we need here is the case when $A = \mathbb{C} \setminus \{0\}$ and $S$ is 1-1:

If $S$ is 1-1 then $E_S(\mathbb{C} \setminus \{0\}) = \bigcap_{n=1}^{\infty} S^n Y.$

[Proof. If $y \in \cap S^n Y$ and $y = S^n y_n$, $n = 1, 2, \ldots$ then $y_1 = Sy_2 = S^2 y_3 = \cdots$ so $y_1 \in \cap S^n Y.$]

Once we recall that $\lambda \in \mathbb{C}$ is a critical eigenvalue for $(S, T)$ provided $\lambda$ is an eigenvalue for $S$ and $(T - \lambda)X$ is of infinite codimension in $X$, we are able to understand the statement of

**THEOREM A.** If there is a countable set $G \subseteq \mathbb{C}$ for which $E_S(\mathbb{C} \setminus G) = \{0\}$ then every $\theta \in \mathcal{I}(S, T)$ is continuous if and only if $(S, T)$ has no critical eigenvalues in $G$.

**Sketch of Proof.** The eigenvalue condition is easily seen to be necessary. We indicate the line of attack in proving sufficiency. To simplify slightly, suppose $S \theta = \theta T$. First, by the stability lemma there is a polynomial $p$ with roots in $G$ for which

$$((S - \lambda)p(S) \mathcal{G})^- = (p(S) \mathcal{G})^-$$

for all $\lambda \in G$, where

$$\mathcal{G} = \{y \in Y \mid \exists x_n \to 0 \text{ with } \theta x_n \to y\}.$$  

[Actually, this can be done so as to hold for all $\lambda \in \mathbb{C}$, so countability of $G$ does not really play a role here.] Second, by Mittag-Leffler's theorem there is a dense subspace $W \subseteq p(S) \mathcal{G}$ for which

$$(S - \lambda)W = W, \quad \lambda \in G.$$
[This does depend on countability of \( G \).] By maximality of \( E_S(C \setminus G) \),

\[
W \subseteq E_S(C \setminus G),
\]

hence \( W \), and thereby \( p(S)G \), is \( \{0\} \). Discard all non-eigenvalue roots of \( p \). Thus \( p(T)X \) is of finite codimension, by our assumption of no critical eigenvalues. From this the continuity of \( \theta \) on all of \( X \) follows readily.

**COROLLARY 1.** If \( S \) has countable spectrum and \( E_S(\emptyset) = \{0\} \) then every \( \theta \in \mathcal{I}(S,T) \) is continuous if and only if \( (S,T) \) has no critical eigenvalue.

**Proof.** \( G = \sigma(T) \) and

\[
E_S(C \setminus G) = E_S((C \setminus \sigma(T)) \cap (\sigma(T)))
\]

\[
= E_S(\emptyset) = \{0\}.
\]

This is a good part of the original Johnson–Sinclair result.

An isometry \( S \) for which \( \cap S^n Y = \{0\} \) is called a *semi-shift*. An obvious example is the unilateral right shift.

**COROLLARY 2.** If \( T \in B(X) \) is arbitrary and \( S \in B(Y) \) is a semishift then \( \mathcal{I}(S,T) \) consists of continuous maps.

**Proof.** \( E_S(C \setminus \{0\}) = \{0\} \) and 0 is not an eigenvalue of an isometry.

We shall now extend this last result to arbitrary isometries \( S \) and thus work our way away from the countability condition necessitated by our use of Mittag-Leffler. The cost of this is a mild restriction on \( T \), namely the assumption that \( T \) be decomposable.

Thanks to Ernst Albrecht, [1], we may define \( T \) to be decomposable provided for any open cover \( U \cup V = C \) there are closed \( T \)-invariant subspaces \( X_U, X_V \) for which

\[
\sigma(T | X_U) \subseteq U, \quad \sigma(T | X_V) \subseteq V
\]

and

\[
X = X_U + X_V.
\]
THEOREM B. If $T$ is a decomposable map on the Banach space $X$ and $S$ is an isometry on the Banach space $Y$, then $I(S, T)$ consists entirely of continuous maps if and only if $(S, T)$ has no critical eigenvalue.

The main step in proving $B$ is contained in

PROPOSITION C. Suppose $S \in B(Y)$ is bounded below and satisfies $\cap S^n Y = \{0\}$, and suppose $T \in B(X)$ is decomposable. Then $I(S, T) = \{0\}$.

Proof. As in the proof of Corollary 2, $I(S, T)$ consists of continuous maps. So suppose $\theta \in I(S, T)$. To omit some of the technical details, suppose also that $S\theta = \theta T$. We know that $\ker \theta$ is closed and as $T(\ker \theta) \subseteq \ker \theta$ we may consider $\tilde{T} : X/\ker \theta \to X/\ker T$ defined by $\tilde{T}(x + \ker \theta) := Tx + \ker \theta$. If we let $\theta_1 : X/\ker \theta \to Y$ be defined by $\theta_1(x + \ker \theta) := \theta x$, then $\theta_1 \in I(S, \tilde{T})$.

But $\tilde{T}$ is quasi-nilpotent: let $\varepsilon \in \mathbb{R}_+$ and cover $C : C = \{ |z| < \varepsilon \} \cup (C \setminus \{0\}) = U \cup V$. Then by decomposability we obtain a splitting

$$X = X_U + X_V.$$ 

Since $\sigma(T | X_V) \subseteq V$ we see that

$$X_V \subseteq E_T(C \setminus \{0\}).$$

Moreover, since

$$\theta E_T(C \setminus \{0\}) \subseteq E_S(C \setminus \{0\}) = \{0\}$$

(the inclusion is a consequence of the maximality of $E_S(C \setminus \{0\})$, and since

$$S \theta E_T(C \setminus \{0\}) = \theta T E_T(C \setminus \{0\}) = \theta E_T(C \setminus \{0\}),$$

we get that $X_V \subseteq \ker \theta$, and hence that $X = X_U + \ker \theta$. From this it follows that $\sigma(\tilde{T}) \subseteq U$ and the arbitrariness of $\varepsilon$ shows that $\sigma(\tilde{T}) = \{0\}$.

Suppose $S^{-1} : \text{ran} S \to Y$ has norm $\|S^{-1}\| = \delta_0$. This means that

$$\|S y\| \geq \frac{1}{\delta_0} \|y\| \text{ for every } y \in Y.$$
and hence
\[ \|S^n y\| \geq \frac{1}{\delta_0^n} \|y\| \text{ for every } n. \]

Choose \( n_0 \in \mathbb{N} \) so that
\[ \|\tilde{T}^{n_0}\| < \left( \frac{1}{2\delta_0} \right)^{n_0} \]
and note that
\[
\|\theta_1 x\| \leq \delta_0^{n_0} \|S^{n_0} \theta_1 x\| = \delta_0^{n_0} \|\theta_1 \tilde{T}^{n_0} x\| \\
\leq \delta_0^{n_0} \|\theta_1\| \|\tilde{T}^{n_0}\| \|x\| \leq \left( \frac{\delta_0}{2\delta_0} \right)^{n_0} \|\theta_1\| \|x\| \\
= 2^{-n_0} \|\theta_1\| \|x\|
\]
from which we get the unlikely claim that
\[ \|\theta_1\| \leq 2^{-n_0} \|\theta_1\| , \]
which is only possible with \( \theta_1 = 0 \), hence \( \theta = 0 \).

Now the proof of \( B \) is not difficult.

**Proof of \( B \).** With \( Z := E_2(C \setminus \{0\}) = \cap S^n Y \), \( Z \) is closed and \( S \)-invariant. If we let \( S \) induce \( S_0 : Y/Z \to Y/Z \) then \( S_0 \) is a semi-shift. Moreover, if \( Q : Y \to Y/Z \) is the quotient map then \( Q \theta \in \mathcal{I}(S_0, T) \). Hence, by Proposition C, \( Q \theta = 0 \) so that \( \theta \) maps \( X \) into \( Z \). However, \( S \mid Z \) is an invertible isometry \((S \mid Z \text{ is } 1-1 \text{ and onto } Z) \) so \( \sigma(S \mid Z) \) is a subset of the unit circle \( T \). This means that \( S \mid Z \) has a functional calculus \( (\text{given by } C^\infty(C) \ni f \to f \mid T \to \text{Fourier coefficients } (c_n(f)) \text{ of } f \mid T \to \sum_{n \in Z} c_n(f) T^n) \), so that \( S \mid Z \) is generalized scalar. Since \((S \mid Z, T) \) has no critical eigenvalues, if \((S, T) \) has no critical eigenvalues, the sufficiency of the critical eigenvalue condition follows from known results [3,4,7].

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