MIXED VOLUMES AND CONNECTED VARIATIONAL PROBLEMS

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ABSTRACT. The recent achievements concerning m-curvature equations gave a new point of view to the geometrical theory of mixed volumes of convex bodies which was developed by A.D. Aleksandrov in 1938-1940. A principal goal of this paper is to pose corresponding variational problems correctly and to formulate sufficient conditions for the existence of minimizers.

1. MIXED VOLUMES. Let $X, P$ be two $n$-dimensional Euclidean spaces and $u(x), v(p)$ be a pair of functions from $C^2$. Define mappings $H_u, H_v$ as follows:

$$H_u : X \rightarrow P, \quad p = u_x,$$

$$H_v : P \rightarrow X, \quad x = v_p.$$ 

We have a composition $H_{vu} = H_v \circ H_u$ and

$$H_{vu} : X \rightarrow \hat{X} \subset X.$$ 

The mapping $H_{vu}$ generates an exterior $n$-form $\tilde{\omega}_n = d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n$. We mix this one with $\omega_n = dx^1 \wedge \ldots \wedge dx^n$ as was done in [1]

$$\omega_{m,n-m}[v; u] = \frac{1}{\binom{n}{m}} \sum \sigma(i) d\tilde{x}^{i_1} \wedge \ldots \wedge d\tilde{x}^{i_m} \wedge dx^{i_{m+1}} \wedge \ldots \wedge dx^{i_n} \quad (1)$$

where $\sigma(i)$ is 1 or $-1$ in accordance with the transposition $i = (i_1 \ldots i_n)$ being even or odd and $i_1 < \ldots < i_m, i_{m+1} < \ldots < i_n$.

The exterior $n$-form (1) may be written in a more compact form as

$$\omega_{m,n-m}[v; u] = \mu_m^v[u] \omega_n. \quad (2)$$

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The operator $\mu_m^u[u]$ defined by (2) is a differential operator of the second order generated by $u, v$. As $\omega_{m,n-m}$ is a volume in some sense, it is natural to call $\mu_m^u[u]$ an operator density and to consider such $u, v$ that $\mu_m^u[u] \geq 0$.

Examples. (i) $v = \frac{1}{2} p^2$, $u(x)$ is any function from $C^2$. Then

$$\mu_m^u[u] \equiv \mu_m^D[u] = \frac{1}{(n)} [u_{xx}]_m$$

where $[u_{xx}]_m$ is a sum of all $m$-order minors of the Hessian matrix $u_{xx}$. The set of non-negativity of $\mu_m^D$ contains a cone

$$K_m^D = \{ u \in C^2; [u_{xx}]_i > 0, \ i = 1, \ldots, m \}$$

as a connected component (2). In the case $m = n$ $K_n^D$ coincides with the set of convex functions.

(ii) $v = \sqrt{1 + p^2}$, $u(x)$ is any function from $C^2$. Then

$$\mu_m^u[u] \equiv \mu_m[u] = \frac{1}{(n)} S_m(k)$$

where $S_m(k)$ is an elementary symmetric function of the principal curvatures $k = (k^1 \ldots k^n)$ of the graph $(x, u(x))$. A cone similar to (3) is

$$K_m = \{ u \in C^2; S_i(i) > 0, \ i = 1, \ldots, m \}$$

in this case [2]. As $\mu_m[u]$ has a simple geometrical sense, we shall use sometimes the notation $\mu_m[\Gamma]$ or $\mu_m[\partial \Omega]$ where $\Gamma$, $\partial \Omega$ are surfaces.

(iii) If $v(p)$ is a Lagrange transformation of the convex function $u(x)$

$$v(p) = x^i p^i - u, \ x = v_p,$$

then $\mu_m^u[u] = u$, [1].

2. VARIATIONAL PROBLEMS. As soon as $\mu_m^u[u]$ was interpreted as the density of a measure it was reasonable to consider a family of functionals

$$\mathcal{I}_m^v(u) = \int v(u_x) \omega_{m,n-m}^v = \int_{\Omega} v(u_x) \mu_m^v[u] \ dx .$$

Paper [1] contains the following assertion.
Theorem 1. An equality

\[
\frac{\delta T^v_m}{\delta u} = -(n - m) \mu^u_{m+1}[u]
\]

holds for any \( u, v \in C^2 \).

Therefore an equation

\[
\mu^u_{m+1}[u] = H_m(x, u)
\]  

would be the Euler-Lagrange equation for the functional

\[
I^v_m[u] = T^v_m(u) + (n - m) \int_{\Omega} f(x, u) \, dx
\]

if \( H_m = \partial f / \partial u \). As a particular case we get the \((m + 1)\)-curvature equation corresponding to the generalized area functional

\[
I_m(u) = \int_{\Omega} \left( \sqrt{1 + u_x^2} \mu_m[u] + (n - m) f(x, u) \right) \, dx.
\]  

However the integrands of these functionals depend on the second derivatives of \( u(x) \) and we cannot hope to proceed in the usual way with them. The first obstacle is a mixed type of equation (5) in \( C^2(\Omega) \). But it would be an elliptic type in the cone \( K^v_m = \{ u \in C^2(\bar{\Omega}) ; \mu^v_i[u] > 0, i = 1, \ldots, m \} \) if \( v(p) \) is convex [2]. The second obstacle is boundary conditions. To make the situation clear we shall take as an example the functional (6) and its Euler-Lagrange equation

\[
\mu_{m+1}[u] = H_{m+1}(x, u).
\]

Connect with any \( u \in C^2 \) a set

\[
M_u = \{ v \in C^2(\bar{\Omega}) ; v|_{\partial\Omega} = \varphi(x), v_n - u_n|_{\partial\Omega} \leq 0 \}
\]

where \( \varphi \in C^2(\partial\Omega) \) is a known function, \( v_n \) means a derivative along the inner normal to \( \partial\Omega \) here and further. \( M_u \) contains \( u(x) \) if \( u|_{\partial\Omega} = \varphi(x) \). The following proposition has been proved in [3].
Lemma 2. Let \( u \in C^2(\bar{\Omega}) \) be a minimizer for \( I_m(u) \) on the set \( \mathcal{M}_u \). Assume that \( \partial \Omega \in C^2 \) is a closed surface in \( \mathbb{R}^n \), \( \varphi \in C^2(\partial \Omega) \), \( f \in C^1(\Omega \times R^1) \). Then \( u(x) \) is a solution of the equation (7) and the inequality

\[
\frac{\partial \mu_m[u]}{\partial u_{nn}} \geq 0, \quad x \in \partial \Omega
\]

is fulfilled.

The value \( \frac{\partial \mu_m[u]}{\partial u_{nn}} \) depends on the derivatives of \( \varphi \), \( \partial \Omega \) and \( u_n \) only. Since we may look at (8) as the additional assumption on \( u_n \), it is reasonable to consider (8) as the second boundary condition in some sense. But it is not conditional on \( u(x) \) in the end of ends. For example if \( \varphi(x) = 0 \), then (8) is equivalent to

\[
(-u_n)^{m-1} \mu_{m-1}[\partial \Omega] \geq 0
\]

as it was shown in [4]. Since we have \( u_n \leq 0 \) on \( \partial \Omega \) for any \( u \in K_{m+1} \), \( u|_{\partial \Omega} = 0 \), inequality (9) becomes a condition on the type of boundary.

Lemma 3 contains sufficient conditions for \( \mu \) to be a minimizer [3].

Lemma 3. Let \( u \in K_{m+1} \) be a solution of the equation (7) and \( u|_{\partial \Omega} = \varphi(x) \). Assume that \( \Omega \) is a bounded domain, \( \partial \Omega \), \( \varphi \in C^2 \), \( H_m \in C^1(\Omega \times R^1) \), \( \partial H_m/\partial u \geq 0 \). Then \( u(x) \) gives a strict local minimum to \( I_m \) with \( f = -\int_{\varphi_1}^{\varphi_1} H_m(x, t) \, dt \), \( \varphi_1 = \max_{\partial \Omega} \varphi \), on the set \( \mathcal{M}_u \).

We see that the principal question in this subject is the solvability of the corresponding Dirichlet problem.

3. EXISTENCE THEOREM. The recent achievements in the theory of \( m \)-curvature equations [3]–[9] lead to the following assertion.

Theorem 4. Let \( 1 \leq m \leq n - 1 \), \( \ell \geq 2 \) and \( 0 < \alpha < 1 \). Assume that

(a) \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), \( \partial \Omega \in C^{\ell+2+\alpha} \cap K_m \);
(b) \( \varphi \in C^{\ell+2+\alpha}(\partial \Omega) \);

(c) \( H_m(x, u) \in C^{\ell+\alpha}(\bar{\Omega} \times R^1) \), \( \partial H_m/\partial u \geq 0 \), \( H_m(x, u) \geq \nu > 0 \), and

\[
\int_{\Omega} H_m^{n/m}(x, \varphi_1) \, dx \leq (1 - \chi) \omega_n
\]  

(10)

with some \( \chi > 0 \), \( \omega_n \) being the volume of the unit ball in \( R^n \);

(d) The \( m \)-curvature of \( \partial \Omega \) denoted by \( h_m(x) \) is connected with \( H_m(x, \varphi_1) \) by the inequality

\[
H_m(x, \varphi_1) \leq \frac{n-m}{n} h_m(x), \quad x \in \partial \Omega.
\]  

(11)

Then there exists a unique solution \( u \in C^{\ell+2+\alpha}(\bar{\Omega}) \cap K_m \) of the problem

\[
\mu_m[u] = H_m(x, u), \quad u|_{\partial \Omega} = \varphi(x).
\]  

(12)

**Corollary.** There exists a set \( \mathcal{M}_u \) such that the functional \( I_{m-1} \) achieves its local minimum on \( \mathcal{M}_u \) if the assumptions of theorem 4 are fulfilled for some \( \ell \geq 2 \).

**Remarks.** (i) The inequality (10) was formulated in [4] as a consequence of a sharp condition:

\[
\int_E H_m(x, \varphi_1) \, dx \leq (1 - \chi) \int_{\partial E} \mu_{m-1}(\partial E) \, ds
\]

where \( E \) is any subdomain of \( \Omega \) with \( \partial E \in K_{m-1} \) (see [4]).

(ii) The inequality (11) was discovered by Trudinger [5], [6] as being necessary for solvability of problem (12) with any smooth boundary function \( \varphi(x) \).

Theorem 4 may be proved by combining some results from [4]–[6], [7], [8].
REFERENCES


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