ON SOME TRACE INEQUALITIES

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§1 INTRODUCTION

Let \( A \geq B \geq 0 \) be positive operators on a Hilbert space. It is well-known that this order assumption implies \( \text{Tr}(f(A)) \geq \text{Tr}(f(B)) \), where \( \text{Tr} \) denotes the usual trace and \( f \) is a continuous increasing function on \( \mathbb{R}_+ \) with \( f(0) = 0 \). In fact, singular numbers \( \{\mu_n(\cdot)\}_{n=1,2,...} \) (see [6], [7] for details) satisfy

\[
\mu_n(f(A)) = f(\mu_n(A)) \geq f(\mu_n(B)) = \mu_n(f(B))
\]

because of \( \mu_n(A) \geq \mu_n(B) \) (a consequence of the min-max expression for \( \mu_n(\cdot) \)). Hence, by summing up over \( n \), one obtains the desired estimate.

The purpose of the present note is to point out two generalizations of the above mentioned trace inequality.

§2 RESULTS

Let \( A, B \) be positive operators on a Hilbert space \( H \) satisfying \( A \geq B \geq 0 \). By setting \( q = 2 \) in Furuta's inequality ([5]), we obtain

(1) \[ A^{(p+2r)/2} \geq (A^r B^p A^r)^{1/2} \]

as long as \( p, r \geq 0 \) satisfy

(2) \[ (1 + 2r)2 \geq p + 2r, \quad \text{i.e.}, \quad 2 + 2r \geq p. \]

Extending the continuous linear map

\[
A^{(p+2r)/4} \zeta \in R(A^{(p+2r)/4}) \mapsto (A^r B^p A^r)^{1/4} \zeta \in H
\]

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(well-defined due to (1)), we obtain the contraction $a$ satisfying

$$aA^{(p+2r)/4} = (A^r B^p A^r)^{1/4},$$

(3)

$$a = 0 \text{ on } R(A^{(p+2r)/4})^\perp.$$

From the first equality we easily get

(4)

$$A^r B^p A^r = A^{(p+2r)/4} a^* a A^{(p+2r)/2} a^* a A^{(p+2r)/4}.$$

We claim

(5)

$$A^{(2r-p)/4} B^p A^{(2r-p)/4} = h A^{(p+2r)/2}.$$

with $h = a^* a$, $0 \leq h \leq 1$ (if $2r - p \geq 0$; otherwise we assume the invertibility of $A$ so that the claim trivially follows from (4)). In fact, because the subspace $R(A^{(p+2r)/4}) \oplus \ker A$ is in $H$, it suffices to check

for a vector $\xi = A^{(p+2r)/4} \zeta + \zeta'$ ($\zeta \in (\ker A)^\perp$, $\zeta' \in \ker A$). However, this follows from straight-forward calculations based on (3) and (4).

**THEOREM 1.** Assume $A \geq B \geq 0$ and $p > 1, \alpha \geq \max\{-1, -p/2\}$.

(i) There exists a partial isometry $u$ satisfying

$$A^{\alpha/2} B^p A^{\alpha/2} \leq u^* A^{p+\alpha} u.$$

(ii) For a continuous increasing function $f$ on $\mathbb{R}_+$ with $f(0) = 0$, we have

$$Tr(f(A^{\alpha/2} B^p A^{\alpha/2})) \leq Tr(f(A^{p+\alpha})).$$
In the above statements the invertibility of $A$ is assumed when $\alpha < 0$.

**PROOF.** (i) Let $A^{(p+2r)/4} h = v|A^{(p+2r)/4} h|$ be the polar decomposition. Since

$$|B^{p/2} A^{(2r-p)/4}| = |A^{(p+2r)/4} h| \quad \text{(by (5))}$$

$$= u^* A^{(p+2r)/4} h (h A^{(p+2r)/4} u),$$

we get

$$A^{(2r-p)/4} B^p A^{(2r-p)/4} = u^* A^{(p+2r)/4} h^2 A^{(p+2r)/4} u$$

$$\leq u^* A^{(p+2r)/2} u$$

(recall $0 \leq h \leq 1$). By setting $\alpha = (2r - p)/2 \geq -1$ by (2), but $r$ cannot be negative), we get (i).

(ii) This follows from

$$\mu_n(A^{\alpha/2} B^p A^{\alpha/2}) \leq \mu_n(u^* A^{p+\alpha} u) \leq \mu_n(A^{p+\alpha}),$$

$n = 1, 2, \ldots$. \hfill (Q.E.D.)

It is obvious from the above proof that $u$ in (i) can be chosen to be a unitary when $A, B$ are (finite) matrices. When $0 \leq p \leq 1$, we have $A^p \geq B^p$ (the operator monotonicity of the function $\lambda^p$ on $\mathbb{R}_+$). Therefore, in this case the above (ii) remains valid for any $\alpha \in \mathbb{R}$. Note that (i) says $B^p \leq u^* A^p u$, $p > 1$ (although $B^p \leq A^p$ generally fails). The next fact might also be worth pointing out.

**PROPOSITION 2.** For self-adjoint operators $A, B$ with $A \geq B$, we can find a unitary $v$ satisfying

$$e^B \leq v^* e^A v.$$

**PROOF.** Ando, [1], showed that $A = A^* \geq B = B^*$ guarantees

$$(0 \leq) k = e^{-A/2}(e^{A/2} e^B e^{A/2})^{1/2} e^{-A/2} \leq 1.$$
Let $e^{A/2}k = v | e^{A/2}k|$ be the polar decomposition. (Note that $v$ is a unitary, all the involved operators being invertible.) Since $ke^A k = e^B$, the same argument as in the proof of Theorem 1, (i) shows the desired result. \hfill (Q.E.D.)

The next result will be proved based on a majorization argument.

**THEOREM 3.** Let $A, B$ be positive operators, and $f, g$ be continuous increasing functions on $\mathbb{R}_+$ vanishing at 0. If $A \succeq B$ (or more generally if $\mu_n(A) \geq \mu_n(B)$ for $n = 1, 2, \ldots$, i.e., $A$ spectrally dominates $B$ in the sense of for example [2], [3]), then we get

$$\text{Tr}(f(A)g(A)) \leq \text{Tr}(f(A)^{1/2}g(B)f(A)^{1/2}).$$

**PROOF.** First we further assume $\dim \mathcal{R}(B) = m < +\infty$. Let $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m (> 0)$ be the non-zero eigenvalues of $B$, and $\xi_1, \xi_2, \ldots, \xi_m$ be corresponding (mutually orthogonal) eigenvectors of length 1. Adding some vectors, we obtain an orthonormal basis \{\xi_i\}_{i=1,2,3,...} for $H$. For each $j$ we have

$$\sum_{i=1}^{j} (f(A) \xi_i | \xi_i) \leq \sum_{i=1}^{j} \mu_i(f(A)).$$

In fact, the right hand side always majorizes $\text{Tr}(p f(A) p)$, where $p$ is a projection satisfying $\dim(pH) \leq j$ (see [6], [7]). We now compute

$$\text{Tr}(f(A)^{1/2}g(B)f(A)^{1/2}) = \text{Tr}(g(B)^{1/2}f(A)g(B)^{1/2})$$

$$= \sum_{i=1}^{\infty} (g(B)^{1/2}f(A)g(B)^{1/2} \xi_i | \xi_i)$$

$$= \sum_{i=1}^{m} g(\beta_i)(f(A) \xi_i | \xi_i)$$

$$= g(\beta_m) \sum_{i=1}^{m} (f(A) \xi_i | \xi_i) + \sum_{j=1}^{m-1} (g(\beta_j) - g(\beta_{j+1})) \times \left( \sum_{i=1}^{j} (f(A) \xi_i | \xi_i) \right)$$
\[
\leq g(\beta_m) \sum_{i=1}^{m} \mu_i(A) + \sum_{j=1}^{m-1} (g(\beta_j) - g(\beta_{j+1})) \times (\sum_{i=1}^{j} \mu_i(f(A)))
\]
(by (6) and the decreasingness of \(\{g(\beta_j)\}\))

\[
= \sum_{i=1}^{m} g(\beta_i) \mu_i(f(A))
\]
\[
\leq \sum_{i=1}^{m} g(\mu_i(A)) \mu_i(f(a)) \text{ (because of } \beta_i = \mu_i(B) \leq \mu_i(A))
\]
\[
= \sum_{i=1}^{m} \mu_i(g(A)) \mu_i(f(A))
\]
\[
\leq Tr(f(A)g(A)).
\]

When \(B\) is not necessarily of finite rank, we choose an increasing sequence \(\{p_i\}\) of finite rank projections tending to the identity operator in the strong operator topology. Notice that each finite rank operator \(B_i = p_i B p_i\) is spectrally dominated by \(A\) (because of \(\mu_n(B_i) \leq \mu_n(B) \leq \mu_n(A)\)). Thus the first half of the proof says

\[
Tr(f(A)^{1/2} g(B_i) f(A)^{1/2}) \leq Tr(f(A)g(A)).
\]

Notice that the sequence \(\{f(A)^{1/2} g(B_i) f(A)^{1/2}\}_i\) converges to \(f(A)^{1/2} g(B) f(A)^{1/2}\) in the strong operator topology. Therefore, the lower semi-continuity of \(Tr(\cdot)\) with respect to this topology shows

\[
Tr(f(A)^{1/2} g(B) f(A)^{1/2}) \leq \liminf_{i \to \infty} Tr(f(A)^{1/2} g(B_i) f(A)^{1/2})
\]
\[
\leq Tr(f(A)g(A)). \quad \text{(Q.E.D.)}
\]

All the results in this note remain valid for a semi-finite trace on a von Neumann algebra of type II. (Instead of \(\mu_n(\cdot)\), generalized s-numbers in [4] have to be used.)

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