A CONTINUITY PROPERTY RELATED TO AN INDEX OF NON-WCG AND ITS IMPLICATIONS

Warren B. Moors

Consider a set-valued mapping \( \Phi \) from a topological space \( A \) into subsets of a topological space \( X \). Then \( \Phi \) is said to be upper semi-continuous at \( t \in A \) if given an open set \( W \) in \( X \) containing \( \Phi(t) \) there exists an open neighbourhood \( U \) of \( t \) such that \( \Phi(U) \subseteq W \). For brevity we call \( \Phi \) an usco if it is upper semi-continuous on \( A \) and \( \Phi(t) \) is a non-empty compact subset of \( X \) for each \( t \in A \). If \( X \) is a linear topological space we call \( \Phi \) a cusco if it is upper semi-continuous on \( A \) and \( \Phi(t) \) is a non-empty convex compact subset of \( X \) for each \( t \in A \). An usco (cusco) \( \Phi \) from a topological space \( A \) into subsets of a topological (linear topological) space \( X \) is said to be minimal if its graph does not strictly contain the graph of any other usco (cusco) with the same domain.

For a bounded set \( E \) in a metric space \( X \), the Kuratowski index of non-compactness is
\[
\alpha(E) \equiv \inf \{ r > 0 : E \text{ is covered by a finite family of sets of diameter less than } r \}.
\]
It is well known that if \( X \) is complete then \( \alpha(E) = 0 \) if and only if \( E \) is relatively compact, [6, p.303].

In a recent paper by Giles and Moors [4], a new continuity property related to Kuratowski's index of non-compactness was examined. In that paper they said that a set-valued mapping \( \Phi \) from a topological space \( A \) into subsets of a metric space \( X \) is \( \alpha \) upper semi-continuous at \( t \in A \) if given \( \varepsilon > 0 \) there exists an open neighbourhood \( U \) of \( t \) such that \( \alpha(\Phi(U)) < \varepsilon \). They showed that if the subdifferential mapping of a continuous convex function \( \phi \) on an open convex subset of a Banach space is \( \alpha \) upper semi-continuous on a dense subset of its domain then \( \phi \) is Fréchet differentiable on a dense and \( G_\delta \) subset of its domain. This result led to the consideration of two generalisations of Kuratowski's index of non-compactness.

For a set \( E \) in a metric space \( X \) the index of non-separability is
\[
\beta(E) \equiv \inf \{ r > 0 : E \text{ is covered by a countable family of balls of radius less than } r \},
\]
when \( E \) can be covered by a countable family of balls of a fixed radius, otherwise, \( \beta(E) = \infty \). Further \( \beta(E) = 0 \) if and only if \( E \) is a separable subset of \( X \), [7].

Now, a set-valued mapping $\Phi$ from a topological space $A$ into subsets of a metric space $X$ is said to be $\beta$ upper semi-continuous at a point $t \in A$ if given $\varepsilon > 0$ there exists an open neighbourhood $U$ of $t$ such that $\beta(\Phi(U)) < \varepsilon$. Moors proved that if the subdifferential mapping of a continuous convex function $\phi$ on an open convex subset of a Banach space is $\beta$ upper semi-continuous on a dense subset of its domain, then $\phi$ is Fréchet differentiable on a dense $G_\delta$ subset of its domain.

The second generalisation of Kuratowski's index of non-compactness involves a weak index of non-compactness introduced by de Blasi. Let us denote the closed unit ball $\{x \in X : \|x\| \leq 1\}$ by $B(X)$ and the unit sphere $\{x \in X : \|x\| = 1\}$ by $S(X)$. For a bounded set $E$ in a normed linear space $X$, the weak index of non-compactness is

$$\omega(E) = \inf \{r > 0 : \text{there exist a weakly compact set } C \text{ such that } E \subseteq C + rB(X)\}.$$ For a bounded set $E$ in a Banach space $X$, $\omega(E) = 0$ if and only if $E$ is relatively weakly compact, [3].

A set valued mapping $\Phi$ from a topological space $A$ into subsets of a normed linear space $X$ is said to be $\omega$ upper semi-continuous at $t \in A$, if given $\varepsilon > 0$ there exists an open neighbourhood $U$ of $t$ such that $\omega(\Phi(U)) < \varepsilon$. Giles and Moors [5, Theorem 2.4] showed that if the subdifferential mapping of a continuous convex function $\phi$ on an open convex subset of a Banach space is $\omega$ upper semi-continuous on a dense subset of its domain then $\phi$ is Fréchet differentiable on a dense $G_\delta$ subset of its domain.

We now introduce a new index, which generalises both the $\beta$ index of non-separability, and the $\omega$ weak index of non-compactness.

For a set $E$ in a normed linear space $X$, the index of non-WCG is

$$\gamma(E) = \inf \{r > 0 : \text{there exists a countable family of weakly compact sets}$$

$$\{C_n\}_{n=1}^{\infty} \text{ such that } E \subseteq \bigcup_{n=1}^{\infty} C_n + rB(X)\}.$$ A subset $E$ of a normed linear space is said to be weakly compactly generated if there exists a weakly compact set $C$ such that $E \subseteq \overline{\text{sp}} \{C\}$. 
Proposition 1

For a normed linear space $X$, the index of non-WCG on $X$ satisfies the following properties

1. $\gamma(E) \geq 0$ for any $E \subseteq X$
2. $\gamma(E) = 0$ if and only if $E$ is a weakly compactly generated subset of $X$.
3. $\gamma(E) \leq \gamma(F)$, for $E \subseteq F \subseteq X$.
4. $\gamma\left(\bigcup_{n=1}^{\infty} E_n\right) = \sup\{\gamma(E_n) : n \in \mathbb{N}\}$, where $E_n \subseteq X$ for all $n \in \mathbb{N}$.
5. $\gamma(E) = \gamma(\overline{E})$ for any $E \subseteq X$, where $\overline{E}$ denotes the closure of $E$.
6. $\gamma(E \cap F) \leq \min\{\gamma(E), \gamma(F)\}$, for $E, F \subseteq X$.
7. $\gamma(E+F) \leq \gamma(E) + \gamma(F)$, for $E, F \subseteq X$.
8. $\gamma(kE) = |k| \gamma(E)$, for $E \subseteq X$ and $k \in \mathbb{R}$.
9. $\gamma(\text{co } E) = \gamma(E)$ for $E \subseteq X$ when $X$ is a Banach space, where $\text{co } E$ denotes the convex hull of $E$.

Proof

The proofs of the properties 1. to 9. are straightforward, with the possible exception of 2. and 9. which we now prove.

2. Clearly, if $E$ is weakly compactly generated subset of $X$ then $\gamma(E) = 0$.

Conversely, if $\gamma(E) = 0$ then there exists a sequence of weakly compact sets $\{C_n\}_{n=1}^{\infty}$ such that

$E \subseteq \bigcup_{n=1}^{\infty} C_n$. Let $C \equiv \bigcup_{n=1}^{\infty} \lambda_n^{-1} C_n \cup \{0\}$ where $\lambda_n \equiv \left(\sup\{\|x\| : x \in C_n\} + 1\right) 2^n < \infty$.

We will now show that $C$ is weakly compact. To this end, let $\{W_{\gamma} \subseteq X : \gamma \in \Gamma\}$ be a weak open cover of $C$. So, for some $\gamma_0 \in \Gamma$, $0 \in W_{\gamma_0}$, and in fact for some $m \in \mathbb{N}$ we have that

$2^{-m} B(X) \subseteq W_{\gamma_0}$. Now, $C \setminus W_{\gamma_0} = \bigcup_{n=1}^{m-1} \left(\lambda_n^{-1} C_n \setminus W_{\gamma_0}\right) = \bigcup_{n=1}^{m-1} \lambda_n^{-1} C_n \setminus W_{\gamma_0}$ which is weakly compact (possibly empty). Let $\{W_{\gamma_i} \subseteq X : i \in \{1, 2, \ldots, n\}\}$ be a finite subcover of $C \setminus W_{\gamma_0}$, then $C \subseteq \bigcup_{i=0}^{n} W_{\gamma_i}$. So, indeed $C$ is weakly compact, and for every $n \in \mathbb{N}$ we have that $C_n \subseteq \lambda_n C \subseteq \text{sp}(C)$. 
Therefore, \( E \subseteq \bigcup_{n=1}^{\infty} C_n \subseteq \overline{sp}(C) \) and so \( E \) is a weakly compactly generated subset of \( X \).

9. Clearly, \( \gamma(E) \leq \gamma(\co E) \) by 3., so we prove the reverse inequality. Given \( r > \gamma(E) \) there exists a countable family of weakly compact sets \( \{C_n\}_{n=1}^{\infty} \) such that \( E \subseteq \bigcup_{n=1}^{\infty} C_n + rB(X) \). So
\[
\co E \subseteq \co \left( \bigcup_{n=1}^{\infty} C_n \right) + rB(X) \subseteq \bigcup_{n=1}^{\infty} \co \left( \bigcup_{k=1}^{n} \co C_k \right) + rB(X).
\]
Now \( \co C_k \) is weakly compact for each \( k \in \mathbb{N} \), [2, p.68], so \( \co \left( \bigcup_{k=1}^{n} \co C_k \right) \) is weakly compact for each \( n \in \mathbb{N} \) and then \( \gamma(\co E) \leq r \).

Therefore, \( \gamma(\co E) \leq \gamma(E) \). //

Consider a non-empty bounded subset \( K \) of \( X \). Given \( f \in X^* \setminus \{0\} \) and \( \delta > 0 \), the slice of \( K \) defined by \( f \) and \( \delta \) is the set \( S(K, f, \delta) = \{ x \in K : f(x) > \sup f(K) - \delta \} \). For a set-valued mapping \( \Phi \) from a topological space \( A \) into subsets of a normed linear space \( X \) we say the \( \Phi \) is \( \gamma \) upper semi-continuous at \( t \in A \), if given \( \varepsilon > 0 \) there exists an open neighbourhood \( U \) of \( t \) such that \( \gamma(\Phi(U)) < \varepsilon \).

Before proceeding to the main theorem we need the following two lemmas (see [7, Proposition 3.2]).

**Lemma 2**

Consider an usco (cusco) \( \Phi \) from a topological space \( A \) into subsets of a Hausdorff space (separated linear topological space) \( X \). Then \( \Phi \) is a minimal usco (cusco) if and only if for any open set \( V \) in \( A \) and closed (closed and convex) set \( K \) in \( X \) where \( \Phi(V) \subsetneq K \) there exists a non-empty open subset \( V' \subseteq V \) such that \( \Phi(V') \cap K = \emptyset \).

**Lemma 3**

Let \( A \) be a topological space and \( X \) a Hausdorff space (separated linear topological space). Consider \( \Phi \) a minimal usco (cusco) from \( A \) into subsets of \( X \). Let \( B \) be a closed (closed and convex) subset of \( X \). If for each open subset \( U \) in \( A \), \( \Phi(U) \subsetneq B \) then \( \{ x \in A : \Phi(x) \cap B = \emptyset \} \) is a dense open subset of \( A \).
Theorem 4

Consider a Baire space $\mathcal{A}$, and a Banach space $X$. Let $\tau$ denote either the weak or norm topologies on $X$ or, if $X$ is the dual of a Banach space, also the weak * topology on $X$. Consider a minimal $\tau$-usc (\tau-cusco) $\Phi$ from $\mathcal{A}$ into subsets of $X$. If $\Phi$ is $\gamma$ upper semi-continuous on a dense subset of $\mathcal{A}$ then $\Phi$ is single-valued and norm upper semi-continuous on a dense $G_\delta$ subset of $\mathcal{A}$.

Proof

We will prove the theorem only for the case of minimal $\tau$ cuscos, as the proof for minimal $\tau$ uscs is analogous.

For each $n \in \mathbb{N}$, denote by $U_n$ the union of all open sets $U$ in $\mathcal{A}$ such that the diam $\Phi(U) < \frac{1}{n}$. For each $n \in \mathbb{N}$, $U_n$ is open; we will show that $U_n$ is dense in $\mathcal{A}$. Consider $W$ a non-empty open subset of $\mathcal{A}$. Now there exist a $t \in W$ where $\Phi$ is $\gamma$ upper semi-continuous. So there exists an open neighbourhood $V$ of $t$ contained in $W$ such that $\gamma(\Phi(V)) < \frac{1}{4n}$. Therefore there exists a sequence $\{C_n\}_{n=1}^\infty$ of weakly compact sets in $X$ such that $\Phi(V) \subseteq \bigcup_{k=1}^\infty C_k + \frac{1}{4n}B(X)$.

We now prove that there exist a non-empty open subset $G$ of $V$ such that $\omega(\Phi(G)) < \frac{1}{4n}$. Now if $\Phi(V') \subseteq \overline{\cap} C_1 + \frac{1}{4n}B(X)$ for some non-empty subset $V'$ of $V$, write $G \equiv V'$, but if not, then by Lemma 3 there exists a dense open set $O_1 \subseteq V$ such that $\Phi(O_1) \cap \overline{\cap} C_1 + \frac{1}{4n}B(X) = \emptyset$.

Now if $\Phi(V') \subseteq \overline{\cap} C_2 + \frac{1}{4n}B(X)$ for some non-empty open subset $V'$ of $V$, write $G \equiv V'$, but if not, then by Lemma 3 there exists a dense open set $O_2 \subseteq V$ such that $\Phi(O_2) \cap \overline{\cap} C_2 + \frac{1}{4n}B(X) = \emptyset$.

Continuing in this way we will have defined $G$ at some stage, because if not, $O_\infty \equiv \bigcap_{k=1}^\infty O_k$ is a dense $G_\delta$ subset of $V$ and $\Phi(O_\infty) \cap \left( \bigcup_{k=1}^\infty C_k + \frac{1}{4n}B(X) \right) = \emptyset$. However, for any $t \in V$ we have that $\Phi(t) \cap \left( \bigcup_{k=1}^\infty C_k + \frac{1}{4n}B(X) \right) \neq \emptyset$. So we can conclude that $V$ contains a non-empty open set $G$ with $\omega(\Phi(G)) < \frac{1}{4n}$.
We now prove that there exists a non-empty open subset $U$ of $G$ such that the diameter $\Phi(U) < \frac{1}{n}$. Now there exists a minimal convex weakly compact set $C_m$ such that $\Phi(G) \subseteq C_m + \frac{1}{4n} B(X)$, [5, Lemma 2.2).

We may assume that the diameter $C_m \geq \frac{1}{2n}$. Since $C_m$ is weakly compact and convex there exists an $f \in S(X^*)$ and a $\delta > 0$ such that $\text{diam } S(C_m, f, \delta) < \frac{1}{2n}$, [1, p.199]. Now $K = C_m \setminus S(C_m, f, \delta)$ is a non-empty weakly compact and convex subset of $X$, and so it is $\tau$ closed and convex. But $K + \frac{1}{4n} B(X)$ is also $\tau$ closed and convex. However, since $C_m$ is a minimal convex weakly compact set such that $\Phi(G) \subseteq C_m + \frac{1}{4n} B(X)$ we must have that $\Phi(G) \not\subseteq K + \frac{1}{4n} B(X)$. Since $\Phi$ is a minimal $\tau$ cusco it follows from Lemma 2 that there exists a non-empty open subset $U$ of $G$ such that

$$\Phi(U) \subseteq \left(C_m + \frac{1}{4n} B(X)\right) \setminus \left(K + \frac{1}{4n} B(X)\right) \subseteq S(C_m, f, \delta) + \frac{1}{4n} B(X).$$

So the diameter $\Phi(U) < \frac{1}{n}$, and we have that $\emptyset \neq U \subseteq U_n \cap W$. We conclude that for each $n \in \mathbb{N}$, $U_n$ is dense in $A$ and so $\Phi$ is single-valued and norm upper semi-continuous on the dense $G_\delta$ subset $\bigcap_{n=1}^{\infty} U_n$ of $A$.

Theorem 4 has some important implications in differentiability theory. But first we need the following facts about convex functions. A continuous convex function $\phi$ on an open convex subset $A$ of a Banach space $X$, is said to be Fréchet differentiable at $x \in A$ if $\lim_{t \to 0} \frac{\phi(x+ty) - \phi(x)}{t}$ exists and is approached uniformly for all $y \in S(X)$. A subgradient of $\phi$ at $x_0 \in A$ is a continuous linear functional $f$ on $X$ such that $f(x-x_0) \leq \phi(x) - \phi(x_0)$ for all $x \in A$. The subdifferential of $\phi$ at $x_0$ is denoted by $\partial \phi(x_0)$ and is the set of all subgradients of $\phi$ at $x_0$. The subdifferential mapping $x \to \partial \phi(x)$ is a minimal weak * cusco from $A$ into subsets of $X^*$, [8, p.100]. Further $\phi$ is Fréchet differentiable at $x \in A$ if and only if the subdifferential mapping $x \to \partial \phi(x)$ is single-valued and norm upper semi-continuous at $x$, [8, p.18]. So from Theorem 4, we have the following two corollaries.
Corollary 5

A continuous convex function $\phi$ on an open convex subset $A$ of a Banach space $X$ whose subdifferential mapping $x \rightarrow \partial \phi(x)$ is upper semi-continuous on a dense subset of $A$ is Fréchet differentiable on a dense $G_δ$ subset of $A$.

The well-known property for spaces with weakly compactly generated dual, [8,p.38], follows naturally.

Corollary 6

Every Banach space, whose dual is weakly compactly generated has the property that every continuous convex function on an open convex subset is Fréchet differentiable on a dense $G_δ$ subset of its domain.

References


Department of Mathematics  
University of Newcastle  
NSW 2308, Australia.