Let $X$ be a locally convex Hausdorff space (briefly, lcs) and $L(X)$ be the space of all continuous linear operators of $X$ into itself, equipped with the topology of pointwise convergence in $X$. An element $\xi$ of the dual space $(L(X))'$, of $L(X)$, is the form

$$\xi : T \mapsto \sum_{j=1}^{n} (Tx_j, x'_j), \quad T \in L(X),$$

for some finite subsets $\{x_j\}_{j=1}^{n} \subseteq X$ and $\{x'_j\}_{j=1}^{n} \subseteq X'$. So, the weak topology of the lcs $L(X)$ is the weak operator topology. Despite this simple description, it is often difficult to determine the relative weak compactness of subsets of $L(X)$. However, to determine the relative weak compactness of subsets of the underlying space $X$ may be easier. So, if $\mathcal{A}$ is a subset of $L(X)$, then a natural starting point would be to examine the relative weak compactness, in $X$, of the sets $\mathcal{A}[x] = \{Tx; T \in \mathcal{A}\}, \; x \in X$, and relate this to $\mathcal{A}$ as a subset of $L(X)$. Call a family of operators $\mathcal{A} \subseteq L(X)$ pointwise (relatively) weakly compact whenever the subsets $\mathcal{A}[x], \; x \in X$, of $X$, are (relatively) weakly compact.

**PROPOSITION 1.** Let $\mathcal{A}$ be a subset of $L(X)$.

(i) If $\mathcal{A}$ is relatively weakly compact, then it is also pointwise relatively weakly compact.

(ii) If $\mathcal{A}$ is equicontinuous, then it is relatively weakly compact if, and only if, it is pointwise relatively weakly compact.

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Remark 1. (i) Part (i) follows from the continuity of the map \( T \mapsto Tx, \ T \in L(X) \), which is continuous from \( L(X)_\sigma \) into \( X_\sigma \) (the \( \sigma \) indicating the weak topology), for each \( x \in X \). We shall give another proof whose technique is used later.

(ii) Part (ii) is known, [4; pp.97–98]. It follows, for example, from the following:

(a) Since \( L(X) \subseteq X^X \) (product topology) the weak topology of \( L(X) \) is induced from the product topology of \( (X_\sigma)^X \).

(b) If \( A \subseteq L(X) \) is equicontinuous, then the closure of \( A \) in \( (X_\sigma)^X \) is actually a part of the subspace \( L(X) \).

Such arguments give no real feeling for why such a result “works”. We present a more direct and elementary proof (though somewhat longer).

(iii) There exist relatively weakly compact sets in \( L(X) \) which are not equicontinuous. Indeed, let \( Y = c_0 \) (Banach space) and \( X = Y_\sigma \). Let \( \Sigma \) be the set of all subsets of \( N \) and \( P : \Sigma \to L(X) \) be the \( \sigma \)-additive (spectral) measure of co-ordinate-wise multiplication in \( X \) by elements \( \chi_E, E \in \Sigma \). Since \( L(X) = L(Y) \) as vector spaces, \( P \) can be interpreted as \( L(Y) \)-valued, where it is still \( \sigma \)-additive (by the Orlicz-Pettis lemma). Since \( L(Y) \) is quasicomplete, vector measure theory implies \( A = P(\Sigma) \) is a relatively weakly compact subset of \( L(Y) \) hence, also of \( L(X) \) since \( L(X)_\sigma = L(X) = L(Y)_\sigma \) as lc-spaces. However, \( A \) is not an equicontinuous subset of \( L(X) \).

COROLLARY 1.1. (i) Let \( X \) be a lcs. If there exists a lc-Hausdorff topology \( \rho \) on \( X \) consistent with the duality of \( X \) and \( X' \), such that \( X_\rho \) is barrelled and \( L(X_\rho) \) is equal to \( L(X) \) as a vector space, then a subset of \( L(X) \) is relatively weakly compact if, and only if, it is pointwise relatively weakly compact.

(ii) If \( X \) is a barrelled space, then a subset of \( L(X) \) is relatively weakly compact if, and only if, it is pointwise relatively weakly compact.
(iii) If $X$ is a sequentially complete $(DF)$-space, then a separable subset of $L(X)$ is relatively weakly compact if, and only if, it is pointwise relatively weakly compact.

The proofs of these results will be via a series of lemmas.

If $Y$ is a linear space, then $Y^*$ denotes the algebraic dual space of $Y$ equipped with the topology $\sigma(Y^*, Y)$. If $X$ is a lcs, then $X''$ is the weak completion of $X$. A subset of $X$ is bounded if, and only if, it is bounded as a subset of $X''$. This, together with Theorem 3.2 and Proposition 6.12 of Ch.III in [9], can be used to prove the following

**LEMMA 1.** A subset of $X$ is weakly compact if, and only if, it is bounded and closed in the weak completion $X''$ of $X$.

Let $L(X, X'')$ denote the space of all linear maps from $X$ into $X''$, equipped with the topology of pointwise convergence on $X$. That is, a net $\{T_\alpha\}$ in $L(X, X'')$ converges to an element $T \in L(X, X'')$ if, and only if, $\lim_\alpha \langle T_\alpha x, x' \rangle = \langle Tx, x' \rangle$ for each $x \in X$ and $x' \in X'$. The space $L(X, X'')$ is the weak completion of $L(X)$.

**Proof of Proposition 1(i).** Let $A \subseteq L(X)$ be relatively weakly compact and let $\overline{A}_w$ denote the weak operator closure of $A$ in $L(X)$. Then it suffices to show that $\overline{A}_w[x]$ is compact in $X_\sigma$, for each $x \in X$.

Fix $x \in X$. The boundedness of $\overline{A}_w$ in $L(X)_\sigma$ implies $\overline{A}_w[x]$ is bounded in $X$ and, hence, in $X''$. So, it suffices to show $\overline{A}_w[x]$ is closed in $X''$ (c.f. Lemma 1). If $y$ is in the $X''$- closure of $\overline{A}_w[x]$, then there exists a net $\{T_\alpha x\}$ in $\overline{A}_w[x]$, with each operator $T_\alpha \in \overline{A}_w$, such that $T_\alpha x \to y$ in $X''$. By the weak compactness of $\overline{A}_w$ in $L(X)$ there is a subnet $\{T_\beta\}$ of $\{T_\alpha\}$ and an element $T \in \overline{A}_w$ such that $T_\beta \to T$ in $L(X)_\sigma$. Then $T_\beta x \to Tx$ in $X_\sigma$ and, hence, in $X''$. Since also $T_\beta x \to y$ in $X''$ it follows that $y = Tx$, and so $y \in \overline{A}_w[x]$. This shows that $\overline{A}_w[x]$ is closed in $X''$. ■
**Lemma 2.** Let $A \subseteq L(X)$ be equicontinuous and pointwise relatively weakly compact. If $\overline{A}_w$ denotes the weak operator closure of $A$ in $L(X)$, then $\overline{A}_w$ is equicontinuous, weakly compact and pointwise weakly compact. In fact, if $\overline{A}_x$ denotes the closure of $A[x]$ in $X_\sigma$ then $\overline{A}_w[x] = \overline{A}_x$, for each $x \in X$.

**Proof.** If $T \in \overline{A}_w$ there exists a net $\{T_{\alpha}\} \subseteq A$ such that $T_{\alpha} \rightharpoonup T$ in $L(X)_\sigma$. Let $V$ be any convex, balanced, $\sigma(X,X')$-closed neighbourhood of 0 in $X$. The equicontinuity of $A$ guarantees the existence of a neighbourhood $U$ of 0 in $X$ such that $T_{\alpha}(U) \subseteq V$, for each $\alpha$. Since $V$ is closed in $X_\sigma$ and $T_{\alpha} \rightharpoonup T$ in $L(X)_\sigma$, it follows that $T(U) \subseteq V$. Accordingly, $\overline{A}_w$ is equicontinuous.

Fix $x \in X$. If $y \in \overline{A}_w[x]$, then $y = Tx$ for some $T \in \overline{A}_w$ and hence, there is a net $\{T_{\alpha}\} \subseteq A$ such that $T_{\alpha} \rightharpoonup T$ in $L(X)_\sigma$. In particular, $T_{\alpha}x \rightharpoonup Tx$ in $X_\sigma$ (and so in $X^{**}$ also). Since the net $\{T_{\alpha}x\}$ is contained in the weakly compact set $\overline{A}_x$ it follows from Lemma 1 that the limit $Tx = y \in \overline{A}_x$. This shows that $\overline{A}_w[x] \subseteq \overline{A}_x$, for each $x \in X$.

Being equicontinuous, $\overline{A}_w$ is bounded in $L(X)$ and hence, also in its weak completion $L(X,X^{**})$. So, to show $\overline{A}_w$ is weakly compact it suffices to show it is closed in $L(X,X^{**})$. Let $T$ be in the $L(X,X^{**})$-closure of $\overline{A}_w$ and $\{T_{\nu}\} \subseteq \overline{A}_w$ be a net such that $T_{\nu} \rightharpoonup T$ in $L(X,X^{**})$. Fix $x \in X$. Then $T_{\alpha}x \rightharpoonup Tx$ in $X^{**}$ and, since $\{T_{\alpha}x\} \subseteq \overline{A}_w[x] \subseteq \overline{A}_x$, it follows that $Tx$ belongs to the $X^{**}$-closure of $\overline{A}_x$. Then the weak compactness of $\overline{A}_x$ in $X$ implies that $Tx \in X$ and so $T$ takes its values in $X$ rather than $X^{**}$. If $V$ and $U$ are two neighbourhoods of 0 in $X$ as described above, then a similar argument as used in proving the equicontinuity of $\overline{A}_w$ shows that $T(U) \subseteq V$ and so $T$ actually belongs to $L(X)$. Since $T \in L(X)$ is the limit, in $L(X)_\sigma$, of the net $\{T_{\alpha}\} \subseteq \overline{A}_w$ it follows that $T \in \overline{A}_w$ (as $\overline{A}_w$ is closed in $L(X)_\sigma$). So, $\overline{A}_w$ is closed in $L(X,X^{**})$. 
The inclusions $\tilde{A}_w[x] \subseteq \tilde{A}_x$, $x \in X$, have already been verified. Since $A[x] \subseteq \tilde{A}_w[x]$, it follows that $\tilde{A}_x$ is contained in the $X_\sigma$-closure of $\tilde{A}_w[x]$. But, the weak compactness of $\tilde{A}_w$ in $L(X)$ implies each set $\tilde{A}_w[x]$, for $x \in X$, is compact (hence closed) in $X_\sigma$ (c.f. proof of Proposition 1(i)). So, $\tilde{A}_x \subseteq \tilde{A}_w[x]$ for each $x \in X$.

Proposition 1(ii) now follows immediately from Proposition 1(i) and Lemma 2.

Proof of Corollary 1.1. One direction of part (i) is just Proposition 1(i). Conversely, if $A \subseteq L(X)$ is pointwise relatively weakly compact, then it is a bounded subset of $L(X)$ and hence, also of $L(X_\rho)$. So, $A$ is an equicontinuous part of $L(X_\rho)$. Since $A$ is pointwise relatively weakly compact as a subset of $L(X_\rho)$, Proposition 1(ii) implies that $A$ is relatively weakly compact in $L(X_\rho)$. As the weak operator topologies on $L(X_\rho)$ and $L(X)$ coincide it follows that $A$ is a relatively weakly compact subset of $L(X)$.

(ii) is a special case of (i).

(iii) The sequential completeness of $X$ guarantees that the bounded subsets of $L(X)$ are the same as those when $L(X)$ is equipped with the topology of uniform convergence on the bounded sets of $X$ ([5], p.136, Proposition (8)). Since $X$ is a $(DF)$-space, it then follows that separable, bounded subsets of $L(X)$ are necessarily equicontinuous ([3], p. 166, Corollary 1). The result then follows from Proposition 1(ii), again.

Remark 2. (i) Concerning Corollary 1.1(i), it is well known that if a lcs $X$ has its weak topology, then $\rho = \tau$ (the Mackey topology) has the property that $L(X)$ and $L(X_\rho)$ are equal as linear spaces. Other compatible lc-topologies $\rho$ for which this is the case are also known; see [8], for example. It is also worth noting that part (ii) of Corollary 1.1 is genuinely a special case of (i). For, the space $X$ of Example 4 of [10] is not itself barrelled, but for $\rho = \tau$ the space $X_\rho$ is barrelled (c.f. Proposition 1(i)) and $L(X)$ is equal to $L(X_\rho)$ as a vector space.
(ii) If \(X\) is a Banach space, then we deduce from Corollary 1.1(ii) and Lemma 2 the criterion that \(A \subseteq L(X)\) is weakly compact if, and only if, it is weakly closed and the weak closure of \(A[x]\) is compact in \(X_x, x \in X\) (Ex. 9.2, Ch.VI of [1]).

(iii) Part (iii) of Corollary 1.1 is a different condition than that of (i) and (ii). For, there exist Fréchet spaces whose strong dual space \(X,\) which is necessarily a complete \((DF)\)-space, is not a Mackey space ([12], p. 292) and so cannot be barrelled.

The following definition is a particular case of that given in [6].

A net \(\{T_\alpha\} \subseteq L(X)\) is said to become small on small sets (for the weak topology) if for every neighbourhood \(U\) of 0 in \(X_x\) there is a neighbourhood \(V\) of 0 in \(X_x\) such that for every \(x \in V\) there is \(\alpha_0\) (depending on \(U\) and \(x\)) such that \(T_\alpha x \in U,\) for all \(\alpha \geq \alpha_0.\)

Nets in \(L(X)\) which are either equicontinuous or convergent in the \(L(X_x)\) are necessarily small on small sets (noting that \(L(X)\) is a linear subspace of \(L(X_x)\)).

For certain classes of lcs \(X,\) the notion of nets being small on small sets leads to the following criterion for relative weak compactness in \(L(X)\).

**PROPOSITION 2.** Let \(X\) be a lcs for which \(L(X)\) and \(L(X_x)\) are equal as linear spaces. Then a subset \(A\) of \(L(X)\) is relatively weakly compact if, and only if, it is pointwise relatively weakly compact and has the property that nets in \(\overline{A}_w\) which are Cauchy for the weak operator topology are small on small sets.

**Proof.** If \(A\) is relatively weakly compact in \(L(X)\), then it is also pointwise relatively weakly compact (c.f. Proposition 1(i)). If \(\{T_\alpha\} \subseteq \overline{A}_w\) is Cauchy for the weak operator topology, then the completeness of \(\overline{A}_w\) in \(L(X)_x\) implies there is \(T \in \overline{A}_w\) such that \(T_\alpha \to T\) in \(L(X)_x\) and hence, it follows that also \(T_\alpha \to T\) in \(L(X_x)\). Accordingly, \(\{T_\alpha\}\) is small on small sets.

To prove the converse note that if \(A\) is pointwise relatively weakly compact, then
so is $\overline{A}_w$ (c.f. proof of Lemma 2). So, $\overline{A}_w$ is bounded in $L(X)$ and hence, also in $L(X, X^{**})$. It therefore suffices to show that $\overline{A}_w$ is closed in $L(X, X^{**})$.

Let $T$ be in the $L(X, X^{**})$-closure of $\overline{A}_w$ and $\{T_\alpha\} \subseteq \overline{A}_w$ be a net such that $T_\alpha \to T$ in $L(X, X^{**})$. As in the proof of Lemma 2 it can then be shown that $T$ is $X$-valued rather than $X^{**}$-valued.

Let $U$ be a closed, absolutely convex neighbourhood of 0 in $X_\sigma$. Since $\{T_\alpha\}$ is a Cauchy net for the weak operator topology it is small on small sets (by hypothesis) and hence, there is a neighbourhood $V$ of 0 in $X_\sigma$ such that for each $x \in V$ there is $\alpha_0 = \alpha_0(U, x)$ such that $T_\alpha x \in U, \alpha \geq \alpha_0$. Since $U$ is closed in $X_\sigma$, it follows that $Tx \in U$ whenever $x \in V$, that is, $T(V) \subseteq U$ and so $T \in L(X_\sigma)$. Since $L(X_\sigma)$ equals $L(X)$, as a vector space, $T$ belongs to $L(X)$. But, then $T_\alpha \to T$ in $L(X)_\sigma$ with $\{T_\alpha\} \subseteq \overline{A}_w$, and so, $T \in \overline{A}_w$. This completes the proof.

The result below (i.e. Proposition 3) is a natural extension of the known fact that if $X$ is a Banach space and $A \subseteq L(X)$ is sequentially compact in the weak operator topology, then its weak operator closure is weakly compact; see Exercise 9.4, Ch.VI of [1]. The main ingredient of the proof is the fact that a subset of a metrizable space is weakly compact if, and only if, it is weakly sequentially compact.

D.H. Fremlin introduced a class of topological spaces, called angelic spaces, which have the property that a subset is compact if and only if it is sequentially compact. There are many lcs, including all metrizable spaces, which are angelic for the weak topology. A systematic exposition of such spaces can be found in [2].

As an application of this notion we show that the equicontinuity condition in Proposition 1(ii) cannot be omitted.

Example. Let $X = \ell^1$, equipped with its weak-star topology $\sigma(\ell^1, c_0)$. Then $X$ is a separable, quasicomplete lcs. Let $e^{(n)}, n = 1, 2, \ldots$, be the element of $c_0$ given
by \( e_j^{(n)} = 1 \) for \( 1 \leq j \leq n \) and \( e_j^{(n)} = 0 \) for \( j > n \). Fix any non-zero element \( \xi \in \ell^1 \). Then the sequence \( \{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(X) \) given by \( T_n : x \mapsto \langle x, e^{(n)} \rangle \xi \), for \( x \in X \), converges pointwise in \( X \) to the linear operator \( T \) specified by \( T : x \mapsto \langle x, e \rangle \xi \), for \( x \in X \), where \( e \in \ell^\infty \) is the element given by \( e_j = 1 \), for every \( j = 1, 2, \ldots \). Because \( e \notin c_0 \) it follows that \( T \notin \mathcal{L}(X) \).

Since \( \{T_n x\}_{n=1}^\infty \) converges in \( X \) (to the element \( T x \)), for every \( x \in X \), the set \( \{T_n x\}_{n=1}^\infty \) is relatively compact in \( X \) and hence is relatively weakly compact (as \( X = X_\sigma \)). That is, \( A = \{T_n\}_{n=1}^\infty \) is pointwise relatively weakly compact.

We show that \( A \) is not relatively weakly compact in \( \mathcal{L}(X) \). Noting that the weak operator topology in \( \mathcal{L}(X) \) is the same as the topology of pointwise convergence in \( X \), it suffices to show that \( A \) is not relatively compact in \( \mathcal{L}(X) \). For each \( n = 1, 2, \ldots \), Alaoglu's theorem implies that the set \( K_n = \{ x \in X ; \| x \|_1 \leq n \} \) is compact (hence, countably compact) in \( X \). Moreover, \( X = \bigcup_{n=1}^\infty K_n \). For each \( n = 1, 2, \ldots \), let \( q_n(x) = |x_n| = |\langle x, \varphi_n \rangle| \), for \( x \in X \), where \( \varphi_1 = e^{(1)} \) and \( \varphi_n = e^{(n)} - e^{(n-1)} \), for \( n \geq 2 \). Then \( \{q_n\}_{n=1}^\infty \) is a separating family of continuous seminorms in \( X \) and so generates a metrizable topology coarser than \( \sigma(\ell^1, c_0) \). These facts about \( X \) imply that \( L(X) \) is an angelic lcs (put \( E = F = X \) in (5) on p.40 of [2]). Accordingly, \( A \) is relatively compact in \( L(X) \) if, and only if, it is relatively sequentially compact in \( L(X) \); see the Theorem on p.31 of [2]. But, \( A = \{T_n\}_{n=1}^\infty \) has no convergent subsequence in \( L(X) \) and so is surely not relatively sequentially compact.

\[ \text{PROPOSITION 3.} \quad \text{Let } X \text{ be a lcs such that } X_\sigma \text{ is angelic. If } A \subseteq L(X) \text{ is equicontinuous and sequentially compact for the weak operator topology, then } A \text{ is pointwise weakly compact, } \overline{A}_w \text{ is weakly compact and} \]

\[ (1) \quad \overline{A}_w[x] = A[x], \quad x \in X. \]
In particular, \( A \) is relatively weakly compact.

**Proof.** Fix \( x \in X \). If \( \{x_n\} \) is any sequence in \( A[x] \), then there exists a sequence \( \{T_n\} \subseteq A \) such that \( x_n = T_n x, \ n = 1, 2, \ldots \). The sequential compactness of \( A \) in \( L(X)_\sigma \) implies there exists \( T \in A \) and a subsequence \( \{T_{n(i)}\} \) of \( \{T_n\} \) such that \( T_{n(i)} \to T \) in \( L(X)_\sigma \). So, \( T_{n(i)} x \to Tx \) in \( X_\sigma \). Since \( Tx \in A[x] \), the subsequence \( \{x_{n(i)}\} \) of \( \{x_n\} \) is convergent, in \( X_\sigma \), to an element of \( A[x] \). Hence, \( A[x] \) is sequentially compact in \( X_\sigma \). As \( X_\sigma \) is angelic, it follows that \( A[x] \) is weakly compact in \( X \). Since \( x \in X \) was arbitrary, Lemma 2 implies that (1) holds, and hence, \( A \) is pointwise weakly compact. Lemma 2 then also implies that \( \overline{A}_w \) is weakly compact.

**Remark 3.** It is worth noting that if \( X \) is a separable Fréchet space, then \( L(X) \) is a Suslin space, [11], and so a subset of \( L(X) \) which is weakly compact is necessarily weakly sequentially compact.

A classical result of M. Krein states that in a Banach space \( X \), the convex hull of a relatively weakly compact set is again relatively weakly compact. Krein’s theorem remains valid in any quasicomplete lcs \( X \), but may fail to hold if \( X \) is only sequentially complete; see §2 of [7], for example. Spaces \( X \) for which Krein’s theorem does hold are said to satisfy the convex compactness property for the weak topology, [7]. Example 5 of [10] shows that the lcs \( L(X) \) may not inherit the convex compactness property for the weak topology from the underlying space \( X \). However, if we restrict our attention to the equicontinuous subsets of \( L(X) \) we have the following

**PROPOSITION 4.** Let \( X \) satisfy the convex compactness property for the weak topology. Then the convex hull of any equicontinuous, relatively weakly compact subset of \( L(X) \) is again relatively weakly compact.

**Proof.** Let \( A \subseteq L(X) \) be equicontinuous and relatively weakly compact. Let \( \overline{\sigma}(A) \)
denote the closure, in \( L(X) \), of the convex hull of \( A \). Then \( \overline{co}(A) \) is also equicontinuous and so it suffices to show that \( \overline{co}(A)[x] \) is relatively weakly compact in \( X \), for each \( x \in X \) (c.f. Proposition 1(ii) and Lemma 2). But, for each \( x \in X \), the set \( \overline{co}(A)[x] \) is a subset of the closed convex hull, \( \overline{co}(A[x]) \), of \( A[x] \) in \( X \). Since each set \( A[x], x \in X \), is relatively weakly compact (c.f. Proposition 1(i)), it follows from the convex compactness property of \( X \) that \( \overline{co}(A[x]) \) is weakly compact for each \( x \in X \).

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