ON THE UNIFORM CLASSIFICATION OF $L_p(\mu)$ SPACES

by Anthony Weston *

In this paper we survey results on the uniform classification of $L_p(\mu)$ spaces, we cite several open problems and we tie some loose ends in the existing theory (i.e., Theorem 12(a), (b)).

The topological classification of Banach spaces was initiated in Mazur [17] where he proved

**Theorem 1:** For $1 \leq p, q < \infty$ the real Banach spaces $L_p(0,1)$, $L_q(0,1)$ and $\ell_q$ are homeomorphic.

From Mazur's work it also followed that the unit balls $B(L_p(0,1))$, $B(L_q(0,1))$ and $B(\ell_q)$ are uniformly homeomorphic. We recall that a bijection $f : X \to Y$ between metric spaces is called a uniform homeomorphism if it is uniformly continuous in both directions.

For a thorough study of the topological structure of linear metric spaces we refer the reader to Bessaga and Pelczynski [8]. We would like to mention that [8] includes a proof of the

**Anderson-Kadec Theorem:** Every infinite dimensional, separable, locally convex, complete linear space is homeomorphic to the Hilbert space $\ell_2$.

The papers that led to this theorem are Kadec [13], [14], Anderson [4] and Bessaga and Pelczynski [6], [7]. Note also Torunczyk's generalization in [19]: two Banach spaces are homeomorphic if and only if they have the same density character.

In the present work our interest is in the uniform classification of $L_p(\mu)$ spaces. Combining results of Lindenstrauss [15] and Enflo [10] we get

**Theorem 2:** An infinite dimensional $L_{p_1}(\mu_1)$ is not uniformly homeomorphic to $L_{p_2}(\mu_2)$ if $p_1 \neq p_2$, $1 \leq p_i < \infty$.

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The following two theorems go somewhat in the other direction and, especially, Theorem 4 is of particular contrast to Theorem 2.

**Theorem 3:** (Aharoni [1]) For $1 \leq p \leq 2$, $L_p(0,1)$ (respectively $\ell_p$) is uniformly homeomorphic to a bounded subset of itself.

**Theorem 4:** (Aharoni [1]) For $1 \leq p \leq 2$, $1 \leq q < \infty$, $L_p(0,1)$ is uniformly homeomorphic to a bounded subset of $\ell_q$ (and therefore to a subset of $L_q(0,1)$).

Surprising, then, are the next two theorems. They are due to Aharoni, Maurey and Mityagin and are to be found in [2].

**Theorem 5:** For $2 < p < \infty$, $L_p(0,1)$ (respectively $\ell_p$) is not uniformly homeomorphic to a bounded subset of itself.

**Theorem 6:** For $2 < p < \infty$, $1 \leq q \leq 2$, a $L_p(\mu)$ space is not uniformly homeomorphic to any subset of a $L_q(\mu)$ space.

The following question is still open.

**Open Problem 1:** For $2 < p < \infty$, $2 < q < \infty$, is $L_p(0,1)$ uniformly homeomorphic to a bounded subset of $\ell_q$?

Questions of this genre may be found in Lindenstrauss [15] and Enflo [11].

The following theorem gives examples of uniformly homeomorphic Banach spaces which are not isomorphic.

**Theorem 7:** Let $1 \leq p$, $q$, $p_n < \infty$ be such that $p_n \to p$ then

$$(\sum \oplus \ell_{p_n})_q \text{ is uniformly homeomorphic to } \ell_p \oplus_q (\sum \oplus \ell_{p_n})_q.$$

The case $p = 1$ is due to Ribe [18] and the generalization was obtained by Aharoni and Lindenstrauss [3]. Note that by taking $p = 1$, $q > 1$ and $p_n > 1$ (for all $n$) we obtain a reflexive Banach space which is uniformly homeomorphic to a non-reflexive Banach space.

The following result is due to Enflo (unpublished).

**Theorem 8:** $L_1(0,1)$ and $\ell_1$ are not uniformly homeomorphic.

Enflo's proof used the following basic facts:
1. A uniformly continuous map $T$ from a Banach space into a metric space satisfies a first order Lipschitz condition for large distances i.e., given $\delta > 0$, we can find a $C$ so that
\[ d(Tx, Ty) \leq C\|x - y\| \text{ whenever } \|x - y\| \geq \delta. \]

2. If $x \neq y$ in $L_1(0,1)$ we can find a sequence of metric midpoints $(x_n)$ such that
\[ \|x_j - x_k\| = \frac{1}{2}\|x - y\| \text{ whenever } j \neq k. \]

Using these facts Enflo showed that a uniformly continuous bijection $T : L_1(0,1) \to \ell_1$ will map metric midpoints between (suitably chosen) $x, y$ in $L_1(0,1)$ to “almost” metric midpoints between $Tx, Ty$ in $\ell_1$ and, further, deduced that $T^{-1}$ cannot be uniformly continuous. Benyamini’s survey [5] — on the uniform classification of Banach spaces — includes a proof of Theorem 8.

Bourgain [9] generalizes Enflo’s midpoint argument to obtain

**Theorem 9:** $L_p(0,1)$ and $\ell_p$ are not uniformly homeomorphic for $1 \leq p < 2$.

We have the long standing

**Open Problem 2:** Are $L_p(0,1)$ and $\ell_p$ uniformly homeomorphic for $p > 2$?

So far we have been addressing the classical Banach spaces $L_p(\mu), 1 \leq p < \infty$. We should also like to consider the $F$-spaces $L_p(\mu), 0 < p < 1$. The usual metric on such a space is given by
\[ d(f, g) := \int |f - g|^p d\mu. \]

Theorem 6.2.1 in Enflo [12] says that: if a locally bounded linear space is uniformly homeomorphic to a Banach space with roundness $> 1$, then it is a normable space.

An immediate corollary, for example, is

**Theorem 10:** $L_p(0,1)$ is not uniformly homeomorphic to $L_q(0,1)$ if $0 < p < 1, 1 < q < \infty$.

For $0 < p < 1$ the analysis of uniformly continuous maps out of $\ell_p$ is impaired if one uses the usual metric. In Weston [20], by the introduction of a uniformly equivalent metric on $\ell_p(0 < p < 1)$, Enflo’s midpoint strategy is again exploited to obtain
Theorem 11: For $0 < p, q \leq 1$ the real F-spaces $L_p(0,1)$ and $\ell_q$ are not uniformly homeomorphic.

At this point it is relevant to note that Theorem 1 and the subsequent remark about unit balls is in fact true for all $0 < p, q < \infty$, the understanding being that for $0 < p, q < 1$ we are dealing with F-spaces. In [17] Mazur introduced the bijections

$$M_{p,q} : L_p(0,1) \rightarrow L_q(0,1) : f \mapsto (\text{sign } f)|f|^{p/q}$$

$$m_{p,q} : \ell_p \rightarrow \ell_q : (a_j) \mapsto ((\text{sign } a_j)|a_j|^{p/q})$$

and we have, for example,

Theorem 12: For $0 < p, q < \infty$ the unit balls $B(L_p(0,1))$, $B(L_q(0,1))$ and $B(\ell_q)$ are uniformly homeomorphic. Indeed, we have the following estimates,

(a) For $0 < p < 1 \leq q < \infty$

$$\|M_{p,q}(f) - M_{p,q}(g)\| \leq 2d(f,g)^{1/q} \text{ for all } f,g \in L_p(0,1)$$

whilst

$$d(M_{q,p}(f), M_{q,p}(g)) \leq 2^{q-p}(\frac{q}{p})^p \|f-g\|^p \text{ for all } f,g \in B(L_q(0,1)).$$

(b) For $0 < p \leq q < 1$

$$d(M_{p,q}(f), M_{p,q}(g)) \leq 2^q d(f,g) \text{ for all } f,g \in L_p(0,1)$$

whilst

$$d(M_{q,p}(f), M_{q,p}(g)) \leq 2^{q-p}(\frac{q}{p})^p d(f,g)^{p/q} \text{ for all } f,g \in B(L_q(0,1)).$$

(c) For $1 \leq p \leq q < \infty$

$$\|(M_{p,q}(f) - M_{p,q}(g))\| \leq 2\|f-g\|^{p/q} \text{ for all } f,g \in L_p(0,1).$$

whilst

$$\|(M_{q,p}(f) - M_{q,p}(g))\| \leq (\frac{q}{p})^{q-p} 2^{q-1} \|f-g\| \text{ for all } f,g \in B(L_q(0,1)).$$

Note: The same estimates apply for $m_{p,q}$ (and its inverse $m_{q,p}$).

We should like to give a proof of (a) (the proof of (b) is similar) but first we need to recall three inequalities. The first two are from Mazur [17] and the third is standard.
1. For real numbers $a$ and $b$ and for $t \geq 1$ we have

$$|(\text{sign } a)|a|^{1/t} - (\text{sign } b)|b|^{1/t}| \leq 2^t|a - b|.$$ 

2. For real numbers $a$ and $b$ and for $t \geq 1$ we have

$$|(\text{sign } a)|a|^t - (\text{sign } b)|b|^t| \leq t|a - b|(|a| + |b|)^{t-1}.$$ 

3. If $0 < p < \infty$ then, setting $\gamma_p = \max(1, 2^{p-1})$,

$$|\alpha - \beta|^p \leq \gamma_p(|\alpha|^p + |\beta|^p)$$

for arbitrary (complex) numbers $\alpha$ and $\beta$.

**Proof of Theorem 12(a):** Suppose $0 < p < 1 \leq q < \infty$. Set $t := \frac{q}{p} > 1$. Given $f, g \in L_p(0, 1)$ we see that

$$\|M_{p,q}(f) - M_{p,q}(g)\| = (\int_0^1 |(\text{sign } f)|f|^{p/q} - (\text{sign } g)|g|^{p/q}|^q dx)^{1/q} \leq (\int_0^1 2^q|f - g|^p dx)^{1/q} \text{ by 3.}$$

$$= 2d(f, g)^{1/q}.$$

Given $f, g \in B(L_q(0, 1))$ we see that

$$d(M_{q,p}(f), M_{q,p}(g)) = (\int_0^1 |(\text{sign } f)|f|^{q/p} - (\text{sign } g)|g|^{q/p}|^p dx \leq (\frac{q}{p})^p(\int_0^1 |f - g|^p(|f| + |g|)^{q-p} dx \text{ by 4.})$$

$$\leq (\frac{q}{p})^p(\int_0^1 |f - g|^q dx)^{p/q}(\int_0^1 (|f| + |g|)^q dx)^{q/p}$$

by applying Hölder's inequality with exponent $q/p$. Hence

$$d(M_{q,p}(f), M_{q,p}(g)) \leq (\frac{q}{p})^p\|f - g\|\|f - g\|^{p}(\int_0^1 2^{q-1}(|f|^q + |g|^q) dx)^{\frac{q-p}{q}} \text{ by 5.}$$

$$\leq 2^{q-p}(\frac{q}{p})^p\|f - g\|^q \text{ as } f, g \in B(L_q(0, 1)). \quad \Box$$

In [16] Lövblom studies uniform homeomorphisms between $B(L_p(0, 1))$ and $B(\ell_q)$ for $1 \leq p, q < \infty$ and the next two theorems are from this paper. But first recall that if $X$ and $Y$ are metric linear spaces and $T : B(X) \rightarrow B(Y)$ is a uniform homeomorphism then the modulus of continuity $\delta_T$ is defined by

$$\delta_T(\epsilon) := \sup\{d(Tx, Ty)|d(x, y) \leq \epsilon\}.$$
Theorem 13: Let $1 \leq p < q \leq 2$ and let $T$ be a uniform homeomorphism $B(L_p(0,1)) \to B(L_q(0,1))$, $B(L_p(0,1)) \to B(\ell_q)$ or $B(\ell_p) \to B(\ell_q)$. Then there is a constant $K > 0$ such that
\[
\delta_{T^{-1}}(\delta_T(\varepsilon)) \geq K \varepsilon^{p/q} \quad \text{for all } \varepsilon \leq 1.
\]
Lövblom's proof of Theorem 10 uses the fact that $L_p(\mu)$ has roundness $p$ for $1 \leq p \leq 2$ and hence does not generalise to $p > 2$ or $0 < p < 1$.

Open Problem 3: Can Theorem 13 be established for all $0 < p, q < \infty$.

Theorem 14: Let $1 \leq p < q < \infty$ and let $T$ be $M_{p,q}$ or $m_{p,q}$ restricted to the appropriate unit ball. Then there exists a constant $K > 0$ such that
\[
\delta_{T^{-1}}(\delta_T(\varepsilon)) \leq K \varepsilon^{p/q} \quad \text{for all } \varepsilon \leq 1.
\]

Note that from the estimates in Theorem 12 it is clear that Theorem 14 holds for all $0 < p < q < \infty$. Note also that Theorem 13 is sharp in the case of uniform homeomorphisms $B(L_p(0,1)) \to B(L_q(0,1))$ or $B(\ell_p) \to B(\ell_q)$, $1 \leq p < q \leq 2$, as a result of the Theorem 12(c) estimates on the Mazur maps.

We conclude this paper with two more open problems.

Open Problem 4: Are $\ell_p$ and $\ell_q$ uniformly homeomorphic for $0 < p < q < 1$?

Open Problem 5: Are $L_p(0,1)$ and $L_q(0,1)$ uniformly homeomorphic for $0 < p < q < 1$?

References:


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