NON-DIFFERENTIABLE INVEX

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Abstract. It is well known that various properties of constrained optimization, such as converse Karush-Kuhn-Tucker and duality, remain valid when convex hypotheses are much relaxed, e.g. to invex. But convex does not need derivatives, whereas invex does. However, there is a property intermediate between convexifiable (by transformation of the domain) and invex, which gives a nondifferentiable extension of invex. Its properties will be surveyed.

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1. Introduction. This survey describes the relations between invex functions and some other related functions, namely functions convexifiable by a diffeomorphism of the domain space, and an intermediate class of protoconvex functions, which give an invex analog of nondifferentiable convex functions. Protoconvex functions satisfy a basic alternative theorem, from which follow necessary and sufficient conditions for a class of constrained optimization problems. Under some restrictions, a local protoconvex property follows from invex. Jeyakumar and Mond's V-invex generalization of invex is shown to relate to a scaling of a constraint system.

A differentiable vector function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^k \) is invex if

\[
(\forall x, p) \quad F(x) - F(p) \geq F'(p)\eta(x, p),
\]

defining \( \geq \) by an order cone \( K \subseteq \mathbb{R}^k \). For the minimization problem:

\[
\text{MIN } f(x) \text{ subject to } -g(x) \in S,
\]

let \( f = (f, g) \) and \( K := \mathbb{R}_+ \times S \) (or \( K := Q \times S \)) if \( f \) is vector-valued, and MIN denotes weak minimum with order cone \( Q \). It is well known [6] that the invex property makes necessary Karush-Kuhn-Tucker (KKT) conditions sufficient for a minimum, and also suffices for duality results. Derivatives can be relaxed to Clarke differentials for Lipschitz functions.

Now \( F \) is convex if \( \eta(x, p) = x - p \), and a convex function need not be differentiable. There are several variants of invex that do not require derivatives. Current progress is described. With suitable definitions,

\[
\begin{array}{ccc}
\text{(without derivatives)} & \quad & \text{(with derivatives)} \\
\text{convexifiable} & \Rightarrow & \text{protoconvex} \quad \Rightarrow \quad \text{invex} \\
\downarrow & \quad & \downarrow \\
\text{Basic Alternative Theorem} & \quad & \text{Converse KKT} \\
\downarrow & \quad & \\
\text{Necessary & Sufficient Lagrangian Conditions}
\end{array}
\]

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2. Main Definitions and Results. \( F \) is convexifiable if \( H := F \circ \phi^{-1} \) is convex, for some invertible transformation \( \phi \). From \( H \) convex, for \( 0 < \alpha < 1 \),

\[
(1 - \alpha)F(p) + \alpha F(x) = (1 - \alpha)H(\phi(p)) + \alpha H(\phi(x)) \\
\geq H((1 - \alpha)\phi(p) + \alpha \phi(x)) \\
= F(\xi(\alpha, x, p)),
\]

where

\[
\xi(\alpha, x, p) := \alpha^{-1}((1 - \alpha)\phi(p) + \alpha \phi(x)) \\
= (1 - \alpha)p + \alpha x \text{ if } F \text{ is convex}.
\]

If \( \phi \) is differentiable, then there exists

\[
\left(\frac{\partial}{\partial \alpha}\right)\xi(\alpha, x, p)_{\alpha=0} = \phi^{-1}'(\phi(p))[\phi(x) - \phi(\alpha)] \equiv \eta(x, p) \tag{2.2}
\]

The combination of the convexlike property (2.1) (see [7]) with (2.2) has been called protoconvex (see [5], also [4] where it was called miniconvex).

If \( F \) is also differentiable, then invex follows from protoconvex by letting \( \alpha \to 0 \) in (2.2) If \( F \) is Lipschitz, then \( F'(p)\eta(x, p) \) is replaced by Clarke's generalized directional derivative \( F^0(p, \eta(x, p)) \), [1].

From (2.1) there follows the Basic Alternative Theorem [7] (see also 2, [3]) for a convexlike function \( F : \Gamma \to Y \), where \( \Gamma \) is convex, and an ordering defined by a closed convex cone in \( Y \), namely :

\[
(\exists x \in \Gamma) \ F(x) < 0 \Rightarrow (\exists \rho \neq 0) \rho F(\Gamma) \geq 0.
\]

Applied to problem (1.2), with \( intS \neq \emptyset \), and \( f(p) = 0 \), it gives :

\[
\text{MIN at } p \iff F(x) \notin -intK \iff (\exists \rho \neq \in K^*) \rho F(.) \geq 0.
\]

So \( (\tau f + \lambda g)(.) \geq 0 \), with \( \tau \neq 0 \) if a constraint qualification (such as Slater's : \( (\exists x_0) - g(x_0) \in intS \)) is assumed.

If \( f \) and \( g \) are Lipschitz, then Wolfe's dual problem is:

\[
\text{MAX } f(u) + vg(u) \text{ such that } u \in S^*, (f + vg)^0(u, .) \geq 0. \tag{2.3}
\]

Then weak duality follows from protoconvex, since

\[
(f + vg)(x) - (f + vg)(u) \geq (f + vg)^0(u; \eta(x, u)) \geq 0
\]

if \( x \) is feasible for (1.2), and \( u, v \) is feasible for (2.3), so that \( f(x) \geq f(u) + vg(u) \).

3. Relation of invex to protoconvex.

Proposition 1. Let \( F \in C^2 \) be invex at \( p \) with \( C^2 \) scale function \( \eta \). If quadratic terms dominate higher-order terms, then \( F \) is protoconvex near \( p \).

Proof. By shift of origin, \( p = 0 \) and \( F(p) = 0 \) may be assumed. Then the invex property is expressed by \( (\forall x)F(x) \geq F'(0)\eta(x, 0) \). It is required to prove that

\[
F(x) \geq F'(0)\eta(x, 0) \Rightarrow (\forall \alpha \in (0, 1)) \alpha F(x) \geq F(\xi(\alpha, x, 0)).
\]
To do this, expand $F(x) = Ax + x^T B x$ and $\eta(x,0) = x + x^T D x$ up to quadratic terms. The dot subscript means a matrix for each component. Then invex requires that $B - AD \succeq 0$, where here $\succeq 0$ for matrices means positive semidefinite. Substituting the trial function

$$\xi(\alpha, x, 0) := \alpha(x + x^T D x) - \alpha^2 x^T D x$$

leads to the requirement that

$$A(x + x^T D x - \alpha x^T D x) + \alpha x^T B x \leq Ax + x^T B x.$$ 

and thus to

$$(\forall \alpha \in (0, 1)) \ (1 - \alpha)(B - AD) \succeq 0$$

which is true from invex.

**Remark 1.** Calculations with quadratic functions can only show that invex holds locally. Unless the functions are positive definite, which gives convexity, the inequalities can only hold in a restricted domain, until the function ‘turns over’.

One approach towards a global property is by a preliminary transformation of the domain, to map it into a local region. By shift of origin, $p = 0$ can be assumed. Choosing polar coordinates $x = (r, \theta)$, where $r = ||x||$ and $\theta$ lies on the unit sphere, a possible transformation of the domain is given by

$$\hat{x} = \kappa(x) \Leftrightarrow \hat{r} = \tanh kr, \hat{\theta} = \theta.$$ 

Suppose that $F$ is a $C^2$ vector function, and $F \circ \kappa^{-1}$ is invex over a local domain (in which quadratic terms dominate). Since invex is invariant to a diffeomorphism of the domain, it follows that $F$ is also invex, over a larger domain.

**4. V-invex.** Jeyakumar & Mond [8] defined a relaxation of invex, called V-invex. In the present notation, a weight function $\beta_j(.) > 0$ is assumed for each constraint $g_j(x) \leq 0$, and the property is:

$$(\forall x)g_j(x) - g_j(p) \geq \beta_j(x)g_j'(p)\eta(x,p).$$

It suffices to assume this for constraints active at $p$. From this, converse KKT readily follows.

However, if the real function $r_j(.) > 0$, then

$$g_j(.) \leq 0 \Rightarrow G_j(.) := r_j(.)g_j(.) \leq 0.$$ 

Thus, given positive functions $r_j$, the constraints $g_j(.) \leq 0$ are equivalent to the constraints $G_j(.) \leq 0$.

Suppose that $g_j(.)$ is invex with scale function $\eta(.,.)$. If $g_j(p) = 0$ then

$$G_j(x) - G_j(p) = G_j(x) = r_j(x)[g_j(x) - g_j(p)]$$

$$\geq r_j(x)g_j'(p)\eta(x,p)$$

$$= [r_j(x)/r_j(p)]G_j'(p)\eta(x,p).$$

Thus $G_j(.)$ is V-invex with weight function $\beta_j(x,p) = r_j(x)/r_j(p).$
REFERENCES