STRICT PROPERTY $(M)$ IN BANACH SPACES

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Abstract. A new property, namely strict property $(M)$, that implies the Opial property is introduced. We discuss relations between this property and some other well known properties. We also prove that Cesaro sequence spaces have strict property $(M)$.

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Key words. fixed point property, Cesaro sequence spaces; property $(M)$; strict property $(M)$.

1. Introduction. Let $(X, \| \cdot \|)$ be a real Banach space, and $X^*$ be the dual space of $X$. Let $B(X)$, $S(X)$ be the closed unit ball and the unit sphere of $X$, respectively.

**Definition 1.1.** A Banach space $X$ has property $(M)$ if whenever $x_n \xrightarrow{u} 0$ then $\psi(x_n)(x) := \limsup_{n \to \infty} \|x_n - x\|$ is a function of $\|x\|$ only.

Property $(M)$ was introduced by Kalton (see [9]). It is an essential ingredient in his characterization of those separable Banach spaces $X$ for which the compact operators $K(X)$ form an $M$-ideal in the algebra of all bounded linear operators, $L(X)$. That is

$$L(X)^* = (K(X)^\perp \oplus V)_1,$$

for some closed subspace $V$.

It was immediately recognized as a prime candidate for a sufficiency condition for the weak fixed point property but it was not until 1997 that García-Falset and Sims (see [6]) proved that a Banach space with property $(M)$ has the weak fixed point property, i.e., every nonexpansive mapping $T$ from a weakly compact and closed set $A \subset X$ into itself has a fixed point in $A$.

Since in a Banach space $X$, property $(M)$ implies that for every weakly null sequence $\{x_n\} \subset X$ and $x \in X$ we have $\psi(x_n)(tx)$ is an increasing function of $t$ on $[0, \infty)$, (see [6]), it is clear that property $(M)$ is equivalent to $x_n \xrightarrow{u} 0$ and $\|u\| \leq \|v\|$ implying that $\limsup_{n \to \infty} \|x_n + u\| \leq \limsup_{n \to \infty} \|x_n + v\|$.

Suppose that $X$ has property $(M)$, $x_n \xrightarrow{u} 0$ and $\|u\| \leq \|v\|$. Choose $\lambda \geq 1$ such that $\|\lambda u\| = \|v\|$ then

$$\limsup_{n \to \infty} \|x_n + v\| = \limsup_{n \to \infty} \|x_n - v\|$$

$$= \limsup_{n \to \infty} \|x_n + \lambda u\| \geq \limsup_{n \to \infty} \|x_n - u\|.$$  

We now introduce strict property $(M)$.

**Definition 1.2.** A Banach space $X$ has strict property $(M)$ if $X$ has property $(M)$ and $\limsup_{n \to \infty} \|x_n + u\| < \limsup_{n \to \infty} \|x_n + v\|$ whenever $x_n \xrightarrow{u} 0$ and $\|u\| < \|v\|$.

Similar conditions on weak* null sequences in $X^*$ are called property $(M^*)$.

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Definition 1.3. A Banach space $X$ is said to have the Opial property if every weakly null sequence $\{x_n\} \subset X$ satisfies

$$\liminf_{n \to \infty} \|x_n\| \leq \liminf_{n \to \infty} \|x_n + x\|$$

for every $x \in X$ ($x \neq 0$), (see [15]).

It is clear that a Banach space with strict property $(M)$ has the Opial property. Hence a Banach space with strict property $(M)$ has weak normal structure. It is well known that $c_0$ fails to have weak normal structure but it has property $(M)$. This means that strict property $(M)$ is essentially stronger than property $(M)$.

Definition 1.4. A Banach space $X$ has property $(M_p)$, ($1 \leq p < \infty$), if

$$\limsup_{n \to \infty} \|x_n + x\|^p = \limsup_{n \to \infty} \|x_n\|^p + \|x\|^p$$

for all $x \in X$, whenever $x_n \overset{u}{\to} 0$. Property $(M_\infty)$ is the requirement that

$$\limsup_{n \to \infty} \|x_n + x\| = \max \{\limsup_{n \to \infty} \|x_n\|, \|x\|\}.$$

for all $x \in X$, whenever $x_n \overset{u}{\to} 0$.

It is clear that a Banach space with property $(M_p)$ ($1 \leq p < \infty$) has strict property $(M)$.

To obtain the weak fixed point property in certain Banach spaces, García-Falset introduced in [4] the following coefficient.

$$R(X) = \sup \left\{ \liminf_{n \to \infty} \|x_n + x\| : \{x_n\} \subset B(X), x_n \overset{u}{\to} 0, x \in B(X) \right\}.$$

He proved that a Banach space $X$ with $R(X) < 2$ has the weak fixed point property (see [4] and [5]). A Banach space $X$ is nearly uniformly smooth (NUS) if for every $\varepsilon > 0$ there is $\eta > 0$ such that if $t \in (0, \eta)$ and $(x_n)$ is a basic sequence in $B(X)$, then there exists $k > 1$ such that $\|x_1 + tx_k\| \leq 1 + t\varepsilon$ (see [17]). A natural generalization of this notion is WNUS whenever it satisfies the above condition with for some $\varepsilon \in (0, 1)$ in place of for every $\varepsilon > 0$ (see [5]).

Recall that a sequence $\{x_n\}$ is said to be an $\varepsilon$-separate sequence for some $\varepsilon > 0$ if

$$sep(\{x_n\}) = \inf \{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$

An important property that implies weak normal structure is the following property. A Banach space $X$ is said to have the uniform Kadec-Klee property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x$ is a weak limit of a norm one $\varepsilon$-separate sequence then $\|x\| < 1 - \delta$.

It is well known that a Banach space $X$ is WNUS if and only if it is reflexive and $R(X) < 2$ (see [4]) and a Banach space $X$ is NUS if and only if it is reflexive and its dual space has the uniform Kadec-Klee property, (see [8]).

Let $l^0$ denote the set of all real sequences and $\mathbb{N}$ the set of natural numbers.
DEFINITION 1.5. A Köthe sequence space is a subspace $X$ of $l^0$ containing an element $x = \{x(i)\} \in X$ with $x(i) > 0$ and such that for every $x \in l^0$ and $y \in X$ with $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$ we have $x \in X$ and $\|x\| \leq \|y\|$.

DEFINITION 1.6. Let $X$ be a Köthe sequence space. We say that the norm is absolutely continuous at $x = \{x(i)\}$ if $\lim_{n \to \infty} \|((0, \cdots, 0, x(n), x(n+1), \cdots))\| = 0$. Let $X_a$ denote the subspace consisting of those elements $x$ at which the norm is continuous. We say that $X$ has absolutely continuous norm if $X_a = X$.

DEFINITION 1.7. A Köthe sequence space $X$ is said to have the Fatou property if for every sequence $\{x_n\} \subset X$ and $y \in X$ satisfying $|x_n(i)| \uparrow |y(i)|$ for all $i \in \mathbb{N}$ we have $\|x_n\| \to \|y\|$.}

A Cesaro sequence space was introduced by J. S. Shue in 1970, (see [8]). It is, for example, useful for the theory of matrix operators. For $1 < p < \infty$, the Cesaro sequence space ($\text{ces}_p$, for short) is defined by

$$
\text{ces}_p = \left\{ x \in l^0 : \|x\| = \left( \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^{n} |x(i)|^p \right)^{\frac{1}{p}} \right\}.
$$

We will subsequently consider some geometric properties of $\text{ces}_p$.

2. Results.

Theorem 2.1. If a Banach space $X$ has property $(M_p)$ ($1 \leq p < \infty$), then $R(X) = 2^\frac{1}{p}$.

**Proof.** For every weakly null sequence $\{x_n\} \subset S(X)$ and $x \in S(X)$, we get

$$
\lim_{n \to \infty} \sup \|x_n - x\|^p = \lim_{n \to \infty} \sup \|x_n\|^p + \|x\|^p = 2.
$$

So,

$$
\liminf_{n \to \infty} \|x_n - x\| \leq \limsup_{n \to \infty} \|x_n - x\| = 2^\frac{1}{p}.
$$

It is clear that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$
\lim_{i \to \infty} \|x_{n_i} - x\| = \lim_{n \to \infty} \|x_n - x\|^p = 2^\frac{1}{p},
$$

so $R(X) = 2^\frac{1}{p} \quad \square$

We recall that a separable Banach space $X$ is ( weakly ) stable ( see [12] ) if for any pair of ( weakly null sequences ) bounded sequence $\{u_n\}$, $\{v_n\}$ in $X$,

$$
\lim_{n \to \infty} \lim_{m \to \infty} \|u_m + v_n\| = \lim_{m \to \infty} \lim_{n \to \infty} \|u_m + v_n\|
$$

whenever either side exists.
Theorem 2.2. Let $X$ be a separable stable Banach space with property $(M)$ and suppose that $X$ contains no copy of $l_1$. Then $X$ has WNUS.

Proof. Since a stable Banach space is weakly sequentially complete (see [12]) and $X$ contains no copy of $l_1$, we conclude that $X$ is reflexive. By Theorem 3.10 in [9], there exists $p \in (1, \infty)$ such that $X$ has property $(M_p)$. Using Theorem 2.1, we obtain that $R(X) < 2$. Hence $X$ has WNUS. 

Corollary 2.1. Let $X$ be a separable stable Banach space with property $(M)$ and suppose that $X$ contains no copy of $l_1$. Then $X$ has the fixed point property.

Theorem 2.3. If $X$ is a weakly stable Banach space with strict property $(M)$ and contains no copy of $l_1$, then $X$ has the uniform Kadec-Klee property.

Proof. Suppose that $X$ fails to have the uniform Kadec-Klee property. Then there exists an $\varepsilon_0 > 0$ such that for any $0 < \delta < 1 - (1 - \varepsilon_0^p)^{\frac{1}{p}}$ there exists a sequence $\{x_n\} \subset S(X)$ with $\text{sep}(\{x_n\}) \geq \varepsilon_0$ and $x \in X$ such that $x_n \overset{w}{\to} x$ and $\|x\| \geq 1 - \delta$.

It is clear that the sequence $\{x_n - x\}$ is weakly null and $\text{sep}(\{x_n - x\}) \geq \varepsilon_0$. By the Bessaga–Pełczynski selection principle we may, without loss of generality, assume that the sequence is a basic sequence. Put $X_0 = \overline{\text{span}}(\{x_n - x\})$. Then $X_0$ is separable, contains no copy of $l_1$ and has strict property $(M)$. Using Proposition 3.9 in [9], there exists $p \in (1, \infty]$ and a normalized weakly null sequence $\{z_n\}$ such that for every $u \in X_0$ and every $\alpha \in \mathbb{R}$,

$$\lim_{n \to \infty} \|u + \alpha z_n\|^p = \|u\|^p + |\alpha|^p$$

when $p < \infty$ or

$$\lim_{n \to \infty} \|u + \alpha z_n\| = \max\{|\|u\||, |\alpha|\}$$

when $p = \infty$.

Since $X_0$ has strict property $(M)$, we have $1 < p < \infty$.

Now let $(\omega_n) \subset X_0$ be any normalized weakly null sequence generating a type. That is, $\|u + \omega_n\|$ exits for all $n \in \mathbb{N}$. Then we can define $\gamma(\alpha, \beta) = \lim_{n \to \infty} \|\alpha u + \beta \omega_n\|$, where $u \in S(X_0)$. Then by weak stability, if $\alpha, \beta \in \mathbb{R}$,

$$\gamma(\alpha, \beta) = \lim_{n \to \infty} \lim_{m \to \infty} \|\alpha z_n + \beta_n \omega_n\|$$

$$= \lim_{m \to \infty} \lim_{n \to \infty} \|\alpha z_n + \beta_n \omega_n\| = (|\alpha|^p + |\beta|^p)^{\frac{1}{p}}.$$

Hence

$$1 = \|x_n\| = \|x + x_n - x\| = \gamma(\|x\|, \|x_n - x\|)$$

$$= (\|x\|^p + \|x_n - x\|^p)^{\frac{1}{p}} \geq ((1 - \delta)^p + \varepsilon^p)^{\frac{1}{p}} > 1,$$
Theorem 2.4. Let $X$ be a separable Banach space and $X^*$ have property $(M^*)$. Then $X$ has strict property $(M)$.

Proof. By Proposition 2.3 in [9], we get that $X$. Next, we will prove that $X$ has strict property $(M)$. For any $u, v \in X$ with $\|u\| > \|v\|$ and every weakly null sequence $(x_n) \subset X$, we pick up $x_n^* \in S(X^*)$ such that $\langle v + x_n, x^* \rangle = \|v + x_n\|$ for all $n \in \mathbb{N}$. By passing to a subsequence we may suppose that $x_n^* \rightharpoonup x^*$. Take $u^* \in S(X^*)$ such that $\langle u, u^* \rangle > \|u\| > \|v\| \geq \langle v, \frac{x^*}{\|x^*\|} \rangle$ and put $\omega^* = \|x^*\| u^*$. Then

$$\limsup_{n \to \infty} \|v + x_n\| = \limsup_{n \to \infty} \langle v + x_n, x^* \rangle$$

$$= \limsup_{n \to \infty} \langle (v, x^*) + (x_n, x_n^* - x^*) \rangle$$

$$< \limsup_{n \to \infty} \langle (u, \omega^*) + (x_n, x_n^* - x^*) \rangle$$

$$= \limsup_{n \to \infty} \langle u + x_n, \omega^* + x_n^* - x^* \rangle$$

$$\leq \limsup_{n \to \infty} \|u + x_n\| \|\omega^* + x_n^* - x^*\|$$

$$= \limsup_{n \to \infty} \|u + x_n\| \|x^* + x_n^* - x^*\|$$

since $\limsup_{n \to \infty} \|\omega^* + x_n^* - x^*\| = \limsup_{n \to \infty} \|x^* + x_n^* - x^*\| = 1$ by property $(M^*)$. Hence

$$\limsup_{n \to \infty} \|v + x_n\| < \limsup_{n \to \infty} \|u + x_n\|.$$

Corollary 2.2. A reflexive Banach space $X$ has strict property $(M)$ if and only if it has property $(M)$.

Theorem 2.5. For the following conditions on the Banach space $X$ we have

$$(1) \Rightarrow (2) \Rightarrow (3).$$

$(1)$ $X$ has strict property $(M)$.

$(2)$ If $x_n \rightharpoonup 0$ then for each $x \in X$ we have $\psi_{(x_n)}(tx)$ is a strictly increasing function of $t$ on $[0, \infty)$.

$(3)$ $X$ satisfies the Opial condition.

Proof. The proof is similar to the proof of proposition 2.1 in [6].
Let $X$ be a Köthe sequence space. We define a new property, namely weak property $(M)$ in $X$ as follows: if $x_n \to 0$ and $u, v \in X$ with $|u(i)| \leq |v(i)|$ then

$$\limsup_{n \to \infty} \|u + x_n\| \leq \limsup_{n \to \infty} \|u + x_n\|.$$ 

**Theorem 2.6.** Let $X$ be a Köthe sequence space with the Fatou property. Then $X$ has weak property $(M)$ if and only if $X$ has absolutely continuous norm.

**Proof.** *Necessity.* Suppose that $X$ does not have an absolutely continuous norm. Then there exists $\varepsilon_0 > 0$ and $x_0 \in S(X)$ such that

$$\left\| \sum_{i=n}^{\infty} x_0(i)e_i \right\| \geq \varepsilon_0$$

for any $n \in \mathbb{N}$, where $e_i = (0, 0, \ldots, 1, 0, \ldots)$. Take $n = 0$. Since $X$ has the Fatou property, there is $n_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{n_1} x_0(i)e_i \right\| \geq \frac{3\varepsilon_0}{4}.$$ 

Notice that

$$\lim_{m \to \infty} \left\| \sum_{i=n_1+1}^{m} x_0(i)e_i \right\| \geq \varepsilon_0,$$

so there exists $n_2 > n_1$ such that

$$\left\| \sum_{i=n_1+1}^{n_2} x_0(i)e_i \right\| \geq \frac{3\varepsilon_0}{4}.$$ 

In this way, we get a sequence $\{n_i\}$ of natural numbers such that

$$\left\| \sum_{j=n_i+1}^{n_{i+1}} x_0(i)e_i \right\| \geq \frac{3\varepsilon_0}{4}, \quad i = 1, 2, \ldots.$$ 

Put $x_i = \sum_{j=n_i+1}^{n_{i+1}} x_0(i)e_i$. Then

(1) $\|x_i\| \geq \frac{3\varepsilon_0}{4}$ for all $i \in \mathbb{N};$

(2) $x_i \to 0$ as $i \to \infty$. It is well known that for any Köthe space $X$ the dual space $X^*$ is isometric to $X'/ \oplus S$, where $S$ is the space of all singular functionals over $X$, i.e., functionals which vanish on the subspace $X_a = \{x \in X : \text{the norm is absolutely continuous at } x\}$ and $X' = \{y \in l^0 : \sum_{i=1}^{\infty} x(i)y(i) < \infty \text{ for all } x \in X\}$. This means that every $f \in X^*$ is uniquely represented in the form

$$f = Ty + \varphi,$$
where \( \varphi \in S \) and for \( y \in X' \) the function \( T_y \) is defined by

\[
T_y(x) = \sum_{i=1}^{\infty} x(i)y(i)
\]

for all \( x \in X \).

Consider \( \sum_{i=1}^{\infty} x(i)y(i) < \infty \). We have

\[
\lim_{i \to \infty} \sum_{j=1}^{\infty} x_n(j)y(j) = \lim_{i \to \infty} \sum_{j=n_i+1}^{n_{i+1}} x(i)y(j) = 0.
\]

Take \( i_0 \in N \) large enough so that \( \left\| \sum_{i=i_0+1}^{\infty} x_0(i)e_i \right\| \leq \frac{4\varepsilon_0}{3} \). Put \( z_0 = \sum_{i=i_0+1}^{\infty} x_0(i)e_i \) and \( z_i = -2x_i \) for all \( i \in N \). Then

\( \|z_i + z_0\| = \|z_0\| \leq \frac{4\varepsilon_0}{3} \) for any \( i \) large enough and \( \|z_i\| = 2\|x_i\| \geq \frac{3}{2}\varepsilon_0 \). This contradicts \( X \) having the weak property \( (M) \).

**Sufficiency.** Let \( \varepsilon > 0 \) be given. For any \( u, v \in X \) with \( |u(i)| \leq |v(i)| \), there exists \( i_0 \in N \) such that \( \left\| \sum_{i=i_0+1}^{\infty} v(i)e_i \right\| < \varepsilon \). For every weakly null sequence \( \{x_n\} \subset X \) there exists \( n_0 \in N \) such that \( \left\| \sum_{i=i_0+1}^{\infty} v(i)e_i \right\| < \varepsilon \). Hence

\[
\left\| x_n + u \right\| = \left\| \sum_{i=1}^{\infty} (x_n(i) + u(i))e_i \right\|
\]

\[
\leq \left\| \sum_{i=1}^{i_0} u(i)e_i + \sum_{i=i_0+1}^{\infty} x_n(i)e_i \right\| + 2\varepsilon
\]

\[
\leq \left\| \sum_{i=1}^{i_0} v(i)e_i + \sum_{i=i_0+1}^{\infty} x_n(i)e_i \right\| + 2\varepsilon
\]

\[
\leq \left\| \sum_{i=1}^{\infty} (x_n(i) + v(i))e_i \right\| + 4\varepsilon = \left\| x_n + u \right\| + 4\varepsilon.
\]

By the arbitrariness of \( \varepsilon \), we get that \( \limsup_{n \to \infty} \left\| x_n + u \right\| \leq \limsup_{n \to \infty} \left\| x_n + v \right\| \).

**Corollary 2.3.** Let \( X \) be a Köthe sequence space with the Fatou property. If \( X \) has strict property \( (M) \), then \( X \) has absolutely continuous norm.
Lemma 2.1. Let \( y, z \in c_{esp} \). Then for any \( \epsilon > 0 \) and \( L > 0 \), there exists \( \delta > 0 \) such that
\[
||y + z||^p - ||y||^p | < \epsilon
\]
whenever \( ||y||^p \leq L \) and \( ||z||^p \leq \delta \) (see [1]).

Theorem 2.7. The Cesaro sequence spaces have property \((M_p)\) \((1 < p < \infty)\).

Proof. Let \( \{x_n\} \) be a weak null sequence and let \( x \in X \). Given \( \epsilon > 0 \). Take
\[
r = \max \left\{ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p, \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |y(i)| \right)^p : n \in \mathbb{N} \right\}.
\]
By lemma 2.1, there exists \( \delta \in (0, \epsilon) \) such that
\[
\left| \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i) + y(i)| \right)^p - \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right| < \epsilon,
\]
whenever \( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < L \) and \( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |y(i)| \right)^p \leq \delta \).

Take \( i_0 \in \mathbb{N} \) such that
\[
\sum_{n=i_0+1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < \delta.
\]
Since \( x_n \overset{w}{\rightarrow} 0 \), there exists \( m_0 \in \mathbb{N} \) such that
\[
\sum_{n=1}^{m_0} \left( \frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p < \delta \text{ whenever } m > m_0.
\]
Hence
\[
||x_m + x||^p = \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x_m(i) + x(i)| \right)^p
\]
\[
= \sum_{n=1}^{i_0} \left( \frac{1}{n} \sum_{i=1}^{n} |x_m(i) + x(i)| \right)^p + \sum_{n=i_0+1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x_m(i) + x(i)| \right)^p
\]
\[
\leq \sum_{n=1}^{i_0} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p + \sum_{n=i_0+1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p + 2\epsilon
\]
\[
\leq \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p + \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p + 4\epsilon.
\]
In same a way, we can get
\[
||x_m - x||^p \geq \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p + \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p - 4\epsilon.
\]
By the arbitrariness of \( \epsilon \), we have
\[
\limsup_{m \to \infty} ||x_m - x||^p = \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p + \limsup_{m \to \infty} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \right)^p,
\]

i.e.,

\[
\lim \sup_{m \to \infty} \|x_m - x\|^p = \|x\|^p + \lim \sup_{m \to \infty} \|x_m\|^p.
\]

The proof is complete. □

REFERENCES


