EXPOSING CONDITIONS IMPLYING UNIFORMITY OF
ROTUNDITY

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Abstract. If every functional which exposes a subset of the unit ball of a Banach space does so uniformly strongly (uniformly weakly) then the space is uniformly rotund (weakly uniformly rotund).

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A normed linear space $X$ is rotund if every point of its unit sphere $S(X)$ is an extreme point of its closed unit ball $B(X)$. The space $X$ is uniformly rotund if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $||x - y|| < \varepsilon$ whenever $||x + y|| \geq 2 - \delta(\varepsilon)$ and $x, y \in S(X)$. $X$ is weakly uniformly rotund if for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ such that $|g(x - y)| < \varepsilon$ whenever $||x + y|| \geq 2 - \delta(\varepsilon, g)$ and $x, y \in S(X)$.

If $X$ is uniformly rotund then for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for every $x \in S(X)$ and $f \in S(X^*)$ with $f(x) = 1$ we have that $S(B(X), f, \delta(\varepsilon)) \subseteq x + \varepsilon B(X)$ where $S(B(X), f, \delta(\varepsilon))$ denotes the slice $\{y \in B(X) : f(y) > 1 - \delta(\varepsilon)\}$. If $X$ is weakly uniformly rotund then for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ such that for every $x \in S(X)$ and $f \in S(X^*)$ with $f(x) = 1$ we have that $S(B(X), f, \delta(\varepsilon, g)) \subseteq x + \{y \in X : |g(y)| < \varepsilon\}$. We show that uniformity of slicing of the ball, apart from rotundity, is sufficient to imply uniform rotundity properties.

For each $f \in S(X^*)$ we will denote by $E_f \equiv \{x \in B(X) : f(x) = 1\}$ and we will say that $f$ exposes $B(X)$ if $E_f \neq \emptyset$. The Bishop-Phelps Theorem guarantees that if $X$ is a Banach space then the set of all functionals in $S(X^*)$ that expose $B(X)$ is dense in $S(X^*)$. Given an $f \in S(X^*)$ that exposes $B(X)$ we will say that $E_f$ is strongly exposed by $f$ if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ so that $S(B(X), f, \delta(\varepsilon)) \subseteq E_f + \varepsilon B(X)$ and that $E_f$ is weakly exposed by $f$ if for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ such that $S(B(X), f, \delta(\varepsilon, g)) \subseteq E_f + \{y : |g(y)| < \varepsilon\}$. Our results are a consequence of the following general considerations.

A set-valued mapping $\Phi$ from a topological space $A$ into subsets of the dual $X^*$ of a normed linear space $X$ is weak* upper semi-continuous $t_0 \in A$ if for each weak* open subset $W$ of $X^*$ such that $\Phi(t_0) \subseteq W$ there exists a neighbourhood $U$ of $t_0$ such that $\Phi(U) \subseteq W$. If $\Phi$ is weak* upper semi-continuous and $\Phi$ has non-empty weak* compact convex images at each point of $A$ then we say that $\Phi$ is a weak* cusco on $A$. Further, $\Phi$ is a minimal weak* cusco on $A$ if its graph does not properly contain the graph of any other weak* cusco on $A$. We use the following characterisation of minimality.

Lemma 1. ([2], Lemma 2.5) A weak* cusco $\Phi$ from a topological space $A$ into subsets of the dual $X^*$ of a normed linear space $X$ is a minimal weak* cusco if and only if for any non-empty open subset $V$ of $A$ and weak* closed convex subset $K$ of $X^*$, with $\Phi(V) \subseteq K$, there exists a non-empty open subset $V_1 \subseteq V$ such that $\Phi(V_1) \cap K = \emptyset$.

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A set-valued mapping \( \Phi \) from a metric space \((A, d)\) into subsets of the dual \(X^*\) of a normed linear space \(X\) is said to be Hausdorff norm upper semi-continuous at \(t_0 \in A\) if for each \(\varepsilon > 0\) there exists a \(\delta(\varepsilon, t_0) > 0\) such that \(\Phi(t) \subseteq \Phi(t_0) + \varepsilon B(X^*)\) for all \(t \in A\) with \(d(t, t_0) < \delta(\varepsilon, t_0)\) and is said to be Hausdorff weak* upper semi-continuous at \(t_0 \in A\) if for each \(x \in S(X)\) and \(\varepsilon > 0\) there exists a \(\delta(\varepsilon, x, t_0) > 0\) such that \(\Phi(t) \subseteq \Phi(t_0) + \{f \in X^* : |f(x)| < \varepsilon\}\) for all \(t \in A\) with \(d(t, t_0) < \delta(\varepsilon, x, t_0)\). We will say that \(\Phi\) is uniformly Hausdorff norm upper semi-continuous on a subset \(D\) of \(A\) if for each \(\varepsilon > 0\) there exists a \(\delta(\varepsilon) > 0\) such that \(\Phi(s) \subseteq \Phi(t) + \varepsilon B(X^*)\) for all \(s, t \in D\) with \(d(s, t) < \delta(\varepsilon)\) and is said to be uniformly Hausdorff weak* upper semi-continuous on \(D\) if for each \(x \in S(X)\) and \(\varepsilon > 0\) there exists a \(\delta(\varepsilon, x) > 0\) such that \(\Phi(s) \subseteq \Phi(t) + \{f \in X^* : |f(x)| < \varepsilon\}\) for all \(s, t \in D\) with \(d(s, t) < \delta(\varepsilon, x)\). Uniformly Hausdorff upper semi-continuous mappings have significant single-valuedness properties, as shown in ([2], Proposition 3.4). 

**Proposition 1.** Given a metric space \((A, d)\) and a normed linear space \(X\), with dual \(X^*\), a minimal weak* cusco \(\Phi\) from \(A\) into subsets of \(X^*\) which is uniformly Hausdorff weak* upper semi-continuous on some dense subset \(D\) of \(A\) is single-valued on \(A\) and for each \(x \in S(X)\) the mapping \(t \mapsto \hat{x}(\Phi(t))\) is uniformly continuous on \(A\). Further, if \(\Phi\) is uniformly Hausdorff norm upper semi-continuous on \(D\) then \(\Phi\) is single-valued and uniformly norm continuous on \(A\).

**Proof.** First we will show that \(\Phi\) is single-valued on \(D\). So let us suppose for the purpose of obtaining a contradiction that \(\Phi\) is not single-valued at \(t_0 \in D\). Then there exist \(f_1, f_2 \in \Phi(t_0), r > 0\) and \(x \in S(X)\) such that \((f_1 - f_2)(x) > 3r > 0\). Consider \(K \equiv \{f \in X^*: f(x) \geq f_1(x) - 2r\}\). Since \(\Phi\) is uniformly Hausdorff weak* upper semi-continuous on \(D\) there exists a \(\delta > 0\) so that \(\Phi(s) \subseteq \Phi(t) + \{f \in X^* : |f(x)| < r\}\) whenever \(s, t \in D\) and \(d(s, t) < \delta\). Now, \(\Phi(B(t_0, \delta)) \subset K\) since \(f_2 \not\in K\) so there exists a non-empty open subset \(V_1\) of \(B(t_0, \delta)\) such that \(\Phi(V_1) \cap K = \emptyset\). Now for any \(t \in V_1 \cap D\) we have that \(f_1 \not\in \Phi(t) + \{f \in X^*: |f(x)| < r\}\). But on the otherhand, \(d(t_0, t) < \delta\), which means that \(f_1 \not\in \Phi(t_0) \subseteq \Phi(t) + \{f \in X^*: |f(x)| < r\}\); which is impossible. Hence \(\Phi\) is single-valued on \(D\). For each \(x \in X\) the mapping \(T_x : D \rightarrow R\) defined by \(T_x(t) = \hat{x}(\Phi(t))\) is uniformly continuous on \(D\) and hence has a uniformly continuous extension \(T_x^*\) to \(A\). It now follows from the weak* upper semi-continuity of \(\Phi\) on \(A\) that \(T_x^*(t) = \hat{x}(\Phi(t))\) for all \(t \in A\). Now, from ([4], Proposition 1.4) we have that \(t \mapsto \hat{x}(\Phi(t))\) is a minimal cusco on \(A\). Therefore for each \(x \in S(X)\) the mapping \(t \mapsto \hat{x}(\Phi(t)) = T_x^*(t)\) is uniformly continuous on \(A\). In particular, this implies that \(\Phi\) is single-valued on \(A\).

In the case when \(\Phi\) is uniformly Hausdorff norm upper semi-continuous on \(D\) we have from the previous argument that \(\Phi\) is single-valued on \(A\) and so the mapping \(\Phi_D : D \rightarrow X^*\) defined by \(\Phi_D(t) = \Phi(t)\) is uniformly norm continuous on \(D\) and hence has a uniformly norm continuous extension \(\Phi^*_D\) to \(A\). It now follows from the weak* continuity of \(\Phi\) on \(A\) that \(\Phi^*_D = \Phi\) and so \(\Phi\) is uniformly norm continuous on \(A\).

We now relate the exposure of subsets of the unit ball of a normed linear space to continuity properties of the subdifferential mapping of the dual norm of the space. Given a normed linear space \(X\), the subdifferential of the norm at \(x \in X\) is the subset \(\partial||x|| \equiv \{f \in B(X^*): f(x) = ||x||\}\). The subdifferential mapping \(x \mapsto \partial||x||\) is a weak* cusco from \(X\) into subsets of \(B(X^*)\).
Lemma 2. Let $f_0 \in S(X^*)$. If $E_{f_0}$ is strongly exposed (weakly exposed) by $f_0 \in S(X^*)$ then the subdifferential mapping $f \mapsto \partial ||f||$ from $X^*$ into subsets of $B(\mathbb{X}^*)$ is Hausdorff norm upper semi-continuous (Hausdorff weak* upper semi-continuous) at $f_0$ and $E_{f_0}$ is weak* dense in $\partial ||f_0||$.

Proof. For each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$S(B(X), f, \delta(\varepsilon)/2) \subseteq S(B(X), f_0, \delta(\varepsilon)) \subseteq E_{f_0} + \varepsilon B(X)$$

for each $f \in X^*$ with $||f - f_0|| \leq \delta(\varepsilon)/2$. Hence by Goldstine's theorem we have that:

$$\partial ||f|| \subseteq \overline{S(B(\mathbb{X}^*), \hat{f}, \delta(\varepsilon)/2)}_{w^*} \subseteq \overline{S(B(\mathbb{X}), \hat{f}, \delta(\varepsilon)/2)}_{w^*} \subseteq \overline{E_{f_0} + \varepsilon B(\mathbb{X}^*)}_{w^*}$$

for each $f \in X^*$ with $||f_0 - f|| < \delta(\varepsilon)/2$. This shows that $f \mapsto \partial ||f||$ is Hausdorff norm upper semi-continuous at $f_0$ and that

$$\partial ||f_0|| \subseteq \overline{E_{f_0} + \varepsilon B(\mathbb{X}^*)}_{w^*}$$

for each $\varepsilon > 0$ which gives the first result. The proof for the case when $E_{f_0}$ is weakly exposed by $f_0$ is similar, except with $\delta(\varepsilon)$ replaced by $\delta(\varepsilon, g)$, $\varepsilon B(X)$ replaced by $\{y \in X : |g(y)| < \varepsilon\}$ and $\varepsilon B(\mathbb{X}^*)$ replaced by $\{F \in \mathbb{X}^* : |\hat{g}(F)| \leq \varepsilon\}$. ∎

For a normed linear space $X$ the restriction of the subderivative mapping $x \mapsto \partial ||x||$ to $S(X)$, is a minimal weak* cusco, ([2], Lemma 3.5).

Lemma 3. Consider a subset $D$ of $S(X^*)$.

(i) If for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ so that for every $f \in D$, $S(B(X), f, \delta(\varepsilon)) \subseteq E_f + \varepsilon B(X)$ then the restriction of the mapping $f \mapsto \partial ||f||$ to $S(X^*)$ is uniformly Hausdorff norm upper semi-continuous on $D$.

(ii) If for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ so that for every $f \in D$, $S(B(X), f, \delta(\varepsilon, g)) \subseteq E_f + \{y \in X : |g(y)| < \varepsilon\}$, then the restriction of the mapping $f \mapsto \partial ||f||$ to $S(X^*)$ is uniformly Hausdorff weak* upper semi-continuous on $D$.

Proof. This follows directly from examining the proof of Lemma 2. ∎

By combining Proposition 1 with Lemma 3 we obtain the following geometrical consequences.

Theorem 1. Consider a dense subset $D$ of $S(X^*)$.

(i) $X$ is uniformly rotund if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for every $f \in D$, $S(B(X), f, \delta(\varepsilon)) \subseteq E_f + \varepsilon B(X)$.

(ii) $X$ is weakly uniformly rotund if for each $g \in S(X^*)$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon, g) > 0$ such that for every $f \in D$, $S(B(X), f, \delta(\varepsilon, g)) \subseteq E_f + \{y \in X : |g(y)| < \varepsilon\}$. 

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Proof. In the first case we see that the restriction of the mapping \( f \mapsto \partial \|f\| \) to \( S(X^*) \) is single-valued and uniformly norm continuous, which implies that the dual norm is uniformly Fréchet differentiable, ([1], p.25) and which gives the result by ([1], p.134). In the second case we see that the mapping \( f \mapsto \partial \|f\| \) is single-valued on \( S(X^*) \) and for each \( g \in S(X^*) \) the mapping \( f \mapsto g(\partial \|f\|) \) on \( S(X^*) \) is uniformly continuous, which implies that the dual norm is uniformly Gâteaux differentiable, ([1], p.25) and which gives the result by ([1], p.63).

As a further application of our theory we establish similar results for a dual space.

**Theorem 2.** Consider a dense subset \( D \) of \( S(X) \).

(i) \( X^* \) is uniformly rotund if for each \( \varepsilon > 0 \) there exists a \( \delta(\varepsilon) > 0 \) such that for every \( x \in D \), \( S(B(X^*), \hat{x}, \delta(\varepsilon)) \subseteq E_\hat{x} + \varepsilon B(X^*) \).

(ii) \( X^* \) is weakly uniformly rotund if for each \( G \in S(X^{**}) \) and \( \varepsilon > 0 \) there exists a \( \delta(\varepsilon, G) > 0 \) such that for every \( x \in D \), \( S(B(X^*), \hat{x}, \delta(\varepsilon, G)) \subseteq E_\hat{x} + \{ f \in X^* : |G(f)| < \varepsilon \} \).

Proof. The proof of (i) follows directly from Proposition 1 and Lemma 3. For the proof of (ii), it follows from Proposition 1 and Lemma 3 that the restriction of the subdifferentiable mapping \( x \mapsto \partial \|x\| \) to \( S(X) \) is single-valued on \( S(X) \). Hence for each \( G \in S(X^{**}) \) and \( \varepsilon > 0 \) there exists a \( \delta(\varepsilon, G) > 0 \) so that for every \( x \in D \), \( \sup \{|G(f-g)| : f, g \in S(B(X^*), \hat{x}, \delta(\varepsilon, G)) \} \leq 2\varepsilon \). Given \( F \in S(X^{**}) \) with \( E_F \neq \emptyset \) consider \( S(B(X^*), F, \delta(\varepsilon, G)) \). For \( f, g \) any two elements of \( S(B(X^*), F, \delta(\varepsilon, G)) \) we have \( [f, g] \cap (1 - \delta(\varepsilon, G))B(X^*) = \emptyset \). Hence, by the strong separation theorem there exists an \( x \in S(X) \) so that \( [f, g] \subseteq S(B(X^*), \hat{x}, \delta(\varepsilon, G)) \). Since \( D \) is dense in \( S(X) \) we may assume that \( x \in D \) and so \( |G(f-g)| \leq 2\varepsilon \); which in particular, implies that \( S(B(X^*), F, \delta(\varepsilon, G)) \subseteq E_F + \{ h \in X^* : |G(h)| \leq 2\varepsilon \} \). The proof now follows from Theorem 1 part (ii).

The interesting aspect of Theorem 2 part(ii) is that it has recently been shown that there are non-reflexive Banach spaces whose dual norms are weakly uniformly rotund, [3].

**REFERENCES**


