Chaotic Vibrations of the Infinite Dimensional Harmonic Oscillator Due to a Self-Excitation Boundary Condition

Part I: Controlled Hysteresis

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§1 Introduction

Consider the motion of a vibrating string whose displacement $w(x,t)$ at location x at time t satisfies

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0, \quad 0 < x < 1, \quad t > 0 \tag{1.1}$$

At the left end $x = 0$, assume the string is fixed:

$$w(0,t) = 0, \quad t > 0 \tag{1.2}$$

At the right end $x = 1$, some force $f(t)$ is acting on the string:

$$w_x(1,t) = f(t), \quad t > 0$$

This force $f(t)$ is assumed to be of the nonlinear velocity feedback type: $f(t) = \alpha w_t(1,t) - \beta w_t^3(1,t),\ t > 0$, yielding

$$w_x(1,t) = \alpha w_t(1,t) - \beta w_t(1,t)^3, \quad \alpha, \beta > 0. \tag{1.3}$$

The energy of the wave equation (1.1) at time t is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[w_x^2(x,t) + w_t^2(x,t)\right] dx$$

Subjected to the boundary conditions (1.2), (1.3), the rate of change of energy is

$$\begin{aligned} \frac{d}{dt}E(t) &= \alpha w_t^2(1,t) - \beta w_t^4(1,t) \\ &= \begin{cases} \geq 0 & \text{if } |w_t(1,t)| \text{ is small} \\ \leq 0 & \text{if } |w_t(1,t)| \text{ is large.} \end{cases} \end{aligned}$$

Let

$$\begin{aligned} & w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 < x < 1 \\ & w_0 \in C^1([0,1]), \quad w_1 \in C^0([0,1]) \end{aligned} \tag{1.4}$$

We use the method of characteristics to treat (1.1) – (1.4). By letting

$$\begin{cases} w_x(x,t) = u(x,t) + v(x,t), \\ w_t(x,t) = u(x,t) - v(x,t), \end{cases} \tag{1.5}$$

The PDE is diagonalized into a *first order hyperbolic system*

$$\frac{\partial}{\partial t}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix}, \quad 0 < x < 1, \quad t > 0. \tag{1.6}$$

The boundary condition at the left end $x = 0$, $w(0,t) = 0$, satisfies

$$w_t(0,t) = 0, \quad \forall t > 0,$$

or

$$u(0,t) - v(0,t) = 0,$$
$$u(0,t) = v(0,t), \quad t > 0, \tag{1.7}$$

while at the right end $x = 1$, we have

$$u(1,t) + v(1,t) = \alpha[u(1,t) - v(1,t)] - \beta[u(1,t) - v(1,t)]^3,$$

or

$$\beta[v(1,t) - v(1,t)]^3 + (1-\alpha)[u(1,t) - v(1,t)] + 2v(1,t) = 0, \quad t > 0. \tag{1.8}$$

The initial conditions become

$$\begin{cases} u_0(x) = \frac{1}{2}[w_0'(x) + w_1(x)] \in C([0,1]), \\ v_0(x) = \frac{1}{2}[w_0'(x) + w_1(x)] \in C([0,1]). \end{cases} \tag{1.9}$$

Equations (1.7)–(1.9) form the set of all initial-boundary data for the PDE (1.6).

From (1.6), since u and v satisfy, respectively,

$$u_t - u_x = 0, \quad v_t + v_x = 0,$$

we have the constancy along characteristics:

$$u(x,t) = \text{constant, along } \; x + t = \text{constant},$$
$$v(x,t) = \text{constant, along } \; x - t = \text{constant}.$$

For example, along a characteristic $x - t = \xi$ passing through the initial horizon $t = 0$, we have

$$v(x,t) = v_0(\xi), \quad \forall (x,t) : x - t = \xi, \quad 0 < \xi < 1.$$

When this characteristic intersects the right boundary $x = 1$ at time τ, we have

$$v(1,\tau) = v_0(\xi); \quad \tau = 1 - \xi.$$

At time $t = \tau$, a nonlinear reflection takes place according to (1.8):

$$u(1,\tau) = F(v(1,\tau)), \tag{1.10}$$

The graph of this mapping, $\{(v(1,\tau), u(1,\tau)) | \tau > 0\}$, after v and u being transposed, is a Poincaré section of the solution set $\mathcal{S} \equiv \{(u(x,t), v(x,t)) | 0 \le x \le 1,\ t > 0\}$ of the PDE system (1.6)–(1.9). Furthermore, iterates of F generate the entire solution set $\mathcal{S}$. Therefore we say that *chaotic vibration occurs* if the Poincaré map F is chaotic as an interval map. In (1.10), $u(1,\tau)$ is determined from $v(1,\tau)$ through solving the cubic equation (1.8). The relation (1.10) is obviously nonlinear. This observation alone is not enough. We must further recognize that the correspondence F *may not even be a single-valued mapping* in general: For

$$\beta(u-v)^3 + (1-\alpha)(u-v) + 2v = 0 : \tag{1.11}$$

(i) When $0 < \alpha < 1$, for each given real value v, there exists a *unique real* solution u satisfying (1.11). Thus $u = F(v)$ is well defined.

(ii) When $\alpha > 0$, there exists $v^* > 0$, depending on α and β, such that for each real v satisfying $|v| < v^*$, there exist *three distinct real solutions* u satisfying (1.11). But for $v \in \mathbf{R}$ satisfying $|v| > v^*$, there corresponds a unique real solution u satisfying (1.11).

In this paper, to handle the nonuniqueness of u for $|v| < v^*$ when $\alpha > 1$, we will choose (out of three branches of u-solutions) a single branch as the solution for $u = F(v)$ later. Physically speaking, such a branch can be chosen only with artificial intervention or special engineering design. Nevertheless, after this choice the nonlinear equation (1.11)is indeed satisfied and the solution uniqueness is effected that is totally mathematically acceptable and justified. The correspondence

$$u(1,\tau) = F(v(1,\tau)), \quad \tau > 0, \tag{1.12}$$

Becomes single-valued and is a well-defined functional relation. We call (1.11) the *controlled hysterectic reflection relation.*

§2 Important Properties of the Hysteresis Reflection Curves

Theorem: Let $0 < \alpha \leq 1$. The map $u = F^2(v)$ has three fixed points 0, $v^+_{p^2}, v^-_{p^2}$. 0 is a repelling fixed point and $v^+_{p^2}, v^-_{p^2}$ attracts each $v \neq 0$.

For the case $\alpha > 1$. Given $v \in \mathbb{R}$, $\alpha > 1$, $\beta > 0$ we consider

$$f(x,v) = \beta x^3 + (1-\alpha)x + 2v.$$

Then $f(x,v)$ has local maximum and local minimum respectively at $x = -x^*$ and x^*, where $x^* = \sqrt{\frac{\alpha-1}{3\beta}}$. For $v^* = \frac{\alpha-1}{3}x^*$, we have

(i) $f(x,v) = 0$ has a unique root $g(v)$ for $|v| > v^*$

(ii) $f(x,v) = 0$ has three distinct real roots $g_1(v) \leq g_2(v) \leq g_3(v)$ for $|v| \leq v^*$.

Thus we define the controlled hysteresis reflection relations.

$$u = F(v) = \begin{cases} v + g(v), & |v| > v^* \\ v + g_2(v), & |v| \le v^* \end{cases}$$
$$= \begin{cases} F_1(v), & v < -v^* \\ F_2(v), & -v* < v < v^* \\ F_3(v), & v > v^* \end{cases}$$

Lemma 1: For $\alpha \ge 7$. The following "Overlapping condition" holds

$$F_3(F_2(v^*)) > F_2(F_3(v^*)).$$

Moreover

$$F_3(F_2(v^*)) - F_2(F_3(v^*)) \to 0 \quad \text{as} \quad \alpha \to \infty.$$

Lemma 2: For $v > v^*$, $0 < F_3'(v) < 1$;

$v < -v^*$, $0 < F_1'(v) < 1$;

$|v| < v^*$, $F_2'(v) > 1$.

Theorem 1: Let $v^* - 2x^* \le v^*_{n_1(\alpha)} < \cdots < v_1^* < v_0^* = v^*$

$I_j = [v_j^*+, v_{j-1}^*-]$, $j = 1, 2 \cdots n_1(\alpha)$. Assume $n_1(\alpha) \ge 3$ (or $7 \le \alpha \le 13.7853$). Then we have a shift sequence

$$I_{n_1(\alpha)} \to I_{n_1(\alpha)-1} \to \cdots \to I_1 \to I_{n_1(\alpha)} \bigcup I_{n_1(\alpha)-1}$$

Consequently by Keener [K] $\rho_{I_{n_1}(\alpha)} \supseteq (0, \frac{1}{n_1(\alpha)}]$ and the map is chaotic.

Remark:

(i) $I_i \to I_j$ iff $F(I_i) \supseteq I_j$ and

(ii) The rotation number $\rho_I(v) = \overline{\lim\limits_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} \chi_I(F^n(v))$ where χ_I is the characteristic function of the interval I.

(iii) $u = F(v)$ is chaotic in Keener sense [K] if range $\rho_I \supseteq [c, d]$ for some $c < d$.

Theorem 2. *If* $3 > n_1(\alpha) \geq 2$ *(or* $\alpha > 13.7853$*) there exists* $k \in \mathbb{Z}^+$ *such that* F^k *is chaotic.*

Proof: We apply the theorem of Malkin [M] to prove the Theorem 2. For details see [CHZ].

Theorem 3. *If* $F_3(v^*) < 0,\ F_3(v^* + x^*) > 0$ *i.e.* $4.5103 < \alpha < 7$ *then* F *is chaotic with a single strangle attractor* $R = [-(v^* + x^*), v^* + x^*] \times [-(v^* + x^*), v^* + x^*]$

Proof: Let $F(\theta) = 0,\ \theta \in (v^*, v^* + x^*)$. Let k be a positive integer s.t. $v_k^* \leq F(-v^{*-}) < v_{k-1}^*$. Define $I_0 = [v^*, v^* + x^*],\ I_1 = [v_{k+2}^{*+}, v_{k+1}^{*-}], \cdots,\ I_{k+2} = [v_1^{*+}, v_0^{*-}],\ I_{k+3} = [v_0^{*+}, \theta^-], I_{k+4} = -I_2, I_{k+3} = -I_3, \cdots, I_{2k+4} = -I_{k+2},\ I_{2k+5} = -I_{k+3}$.

Then we have shift sequence

$$I_1 \to I_2 \to \cdots \to I_{k+2} \to I_{k+3} \to \cdots \to I_{2k+5} \to I_1 \cup I_2$$

By Keener [K], we have

$$\text{Range } \rho_{I_0} \supseteq \left[0, \frac{1}{k+1}\right]$$

and the map is chaotic.

For the case $1 < \alpha \leq 4.5103$, in our paper [CHZ], we show that the map F is periodic when α is small and then becomes transient chaos and finally is chaotic as we increase α.

References

[CHZ] G. Chen, S.B. Hsu and J. Zhou, Chaotic vibrations of the one-dimensional wave equation due to a self-excitation boundary condition. To appear in AMS transaction.

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