ON THE BLOW-UP BEHAVIOR OF SOLUTIONS OF
SCALAR CURVATURE EQUATION AND ITS
APPLICATION

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1. INTRODUCTION

In this expository article, I want to report the recent joint work with 
Chiun-Chuen Chen. Consider positive smooth solutions of the scalar 
curvature equation

\[ \Delta u + K(x)u^{n+2 \over 2} = 0 \quad \text{in } \Omega \subseteq \mathbb{R}^n, \quad (1) \]

where \( \Delta \) is the Laplace operator, \( K(x) \) is a positive \( C^1 \) function and \( n \geq 3 \). Throughout the paper, we always assume that \( K(x) \) is bounded between two positive constants. One of the motivations in studying 
equation (1) arises from the problem of prescribing scalar curvature 
in conformal geometry. Let \((M,g_0)\) be a \( n \)-dimensional Riemannian 
manifold and \( K(x) \) be a given smooth function on \( M \), we would like 
to find a metric \( g \) conformal to \( g_0 \) such that \( K \) is the scalar curvature 
of \( g \). Set \( g = u^{4 \over n-2} g_0 \) for some positive function \( u \), then the problem 
above is equivalent to finding positive smooth solutions of

\[ {n-1 \over 4(n-2)} \Delta_0 u - k_0 u + K(x)u^{n+2 \over n-2} = 0 \quad \text{in } M, \quad (2) \]

where \( \Delta_0 \) denotes the Beltrami-Laplace operator of \((M,g_0)\) and \( k_0(x) \) 
is the scalar curvature of \( g_0 \). When \((M,g_0)\) is the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), then we have \( k_0 \equiv 0 \) and equation (2) reduces to (1) 
after an appropriate scaling.

For the case \( K(x) \equiv a \) a positive constant, say \( K(x) \equiv n(n - 2) \), and 
\( \Omega \equiv \mathbb{R}^n \), all solutions of equation (1) can be completely classified.

**Theorem 1.1.** (Caffarelli-Gidas-Spruck) Any positive smooth solution \( u \) of

\[ \Delta u + n(n-2)u^{n+2 \over 2} = 0 \quad \text{in } \mathbb{R}^n \]
must satisfy
\[ u(x) = \left( \frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n-2}{2}} \]
for some \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^n \).

It is not difficult to see that

(i) The total energy, which is defined by
\[
\int_{\mathbb{R}^n} |\nabla u|^2 = n(n-2) \int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} dx = [n(n-2)]^{1-\frac{n}{2}} S_n^{\frac{n}{2}},
\]
is independent of \( \lambda \). Here \( S_n \) is the Sobolev best constant. And the energy is concentrated in a small neigh of \( x_0 \) (say \( x_0 = 0 \)), i.e., for any \( \delta > 0 \)
\[
\int_{|x| \geq \delta} u^{\frac{2n}{n-2}}(x) dx = O(\lambda^{-\frac{n-2}{2}})
\]
as \( \lambda \to +\infty \).

(ii) Denote \( M = \max_{\mathbb{R}^n} u = \lambda^{\frac{n-2}{2}} \). Then
\[
|u(x)| \leq M^{-1} |x|^{2-n},
\]
i.e.,
\[
\min_{|x| \geq \delta} u = O(M^{-1}).
\]

(iii) Let \( w(r) \equiv u(r)r^{\frac{n-2}{2}} = \left( \frac{\lambda r}{1 + \lambda^2 r^2} \right)^{\frac{n-2}{2}} \). Then \( w(r) \) has a unique critical point in \( r > 0 \), i.e., the maximum point \( r = \lambda^{-1} \). (the property (iii) was first observed by R. Schoen. It is an important notion concerning the blow-up behavior.)

Obviously, the difficulty for studying equation (1) comes from the concentration phenomenon mentioned above. Of course, it is of great interest to study the blow-up behavior of solution of (1) when \( K(x) \) is not a constant function. (or even \( K(x) \equiv \) a constant, but solutions \( u \) is not defined in the whole space \( \mathbb{R}^n \).) In the following sections, we will discuss the blow up behavior and see what is the property of \( K \) affecting the blow-up behavior of a sequence of solutions of (1). Before going into the next section, we would like to point out that a Harnack-type inequality holds for solutions of equation (1) with a constant \( K(x) \).
Theorem 1.2. There exists a constant $c > 0$ such that for any solution $u$ of

$$\Delta u + u^{n+2} = 0 \text{ in } |x| \leq 2R,$$

the inequality,

$$\left(\max_{|x| \leq R} u\right)\left(\min_{|x| \leq 2R} u\right) \leq \frac{c}{R^{n-2}}$$

holds.

Theorem 1.2 was proved in [CLn1], where a more general nonlinear term was considered.

2. SIMPLE BLOW-UPS

Let $u_i$ be a sequence of solutions of equation (1). A point $x_0$ is called a blow-up point if there exists a sequence of $x_i$ such that $x_0 = \lim_{i \to +\infty} x_i$ and $\lim_{i \to +\infty} u_i(x_i) = +\infty$. Following R. Schoen, a blow-up point $x_0$ is called isolated if there exists a local maximum $x_i$ of $u_i$ such that

$$n-2 u_i(x_i) \leq c|x-x_0|^{2-n}$$

for $|x| \leq \delta_0$, (3)

where both constants $c$ and $\delta_0$ are independent of $i$. Note that if $x_0$ is an isolated blow-up point, then $u_i$ is uniformly bounded in any compact set of $B_{\delta_0}(x_0)\setminus\{x_0\}$. Thus we let $M_i = \max_{|x-x_0| \leq \delta_0} u_i(x) = u_i(x_i)$. Obviously, $x_i \to x_0$ as $i \to +\infty$. The blow-up point $x_0$ is called simple if

$$u_i(x_i + x) \leq c|x|^{-\frac{n-2}{2}}$$

for $|x| \leq \delta_0$, (3)

Another notion of the simple blow-up is defined originally by R. Schoen in the following. (See [L1]). Let

$$w_i(r) = \bar{u}_i(r)r^{\frac{n-2}{2}},$$

where $\bar{u}_i(r) = \int_{|x|=r} u$ is the average of $u$ over the sphere $|x| = r$ (for the simplicity of notations, we assume $x_0 = 0$). Then we have

Proposition 2.1. Let $u_i$ be a sequence of solutions of equation (1). Assume that $0$ is an isolated blow-up point of $u_i$. Then $0$ is a simple isolated blow-up point if and only if there exists $r_0 > 0$ such that $w_i(r)$ has a unique critical point in $(0, r_0)$.
Proof. The sufficient part was proved by Y. Y. Li, [L1]. We will give a proof for this part which is different from the one in [L1]. For the proof of Proposition 2.1, we need the following lemma, which can be derived by integrating the differential inequality hold for \( w \). For a proof of Lemma 2.2 below, we refer the reader to [CLn3].

Lemma 2.2. Let \( w(r) \) be defined as in (5) and \( r = e^t \). (The index \( i \) is omitted for the simplicity.) Then

(i) Suppose that \( w \) is nonincreasing in \((t_0, t_1)\) and \( t_1 \) is a local minimum of \( w \), then

\[
\frac{2}{n-2} \log \frac{w(t_0)}{w(t_1)} \leq t_1 - t_0 \leq \frac{2}{n-2} \log \frac{w(t_0)}{w(t_1)} + C. \tag{6}
\]

(ii) Suppose that \( w \) is nondecreasing in \((t_1, t_2)\) and \( t_1 \) is a local minimum of \( w \). Then

\[
\frac{2}{n-2} \log \frac{w(t_2)}{w(t_1)} \leq t_2 - t_1 \leq \frac{2}{n-2} \log \frac{w(t_2)}{w(t_1)} + C, \tag{7}
\]

where \( C \) are a constant depending on \( n \) only.

Return now to the proof of Proposition 2.1.

First, we assume that 0 is a simple blow up point. Let \( T_i < t_i \) denote the first local maximum point and the first local minimum point respectively. Suppose the conclusion of Proposition 2.1 does not hold, i.e., \( \lim_{i \to +\infty} t_i = -\infty \). By a simple argument of scaling, we have

\[
T_i = -\frac{n-2}{2} \log M_i + O(1), \quad \text{and} \tag{8}
\]

\[
\lim_{i \to +\infty} w_i(t_i) = 0. \tag{9}
\]

By (9), we always can find \( t_i^* > t_i \) such that \( w_i(t) \) is increasing in \([t_i, t_i^*]\) and \( t_i^* - t_i \to +\infty \) as \( i \to +\infty \). By (6), (7) and (8), we have

\[
\bar{u}_i (r_i^*) \geq c_1 \bar{u}_i (r_i) \geq c_2 M_i^{-1} r_i^{2-n} \tag{10}
\]

\[
= c_2 (\frac{r_i^*}{r_i})^{n-2} M_i^{-1} r_i^{2-n},
\]
where \( r^*_i = e^{t_i} \) and \( r_i = e^{t_i} \). Since \( \lim_{i \to +\infty} \frac{r^*_i}{r_i} = +\infty \), applying the Harnack inequality, (10) yields a contradiction to (4).

The necessary part follows immediately from the second inequality of (6) and (8).

Q.E.D.

To state our first result, we assume that for any critical point \( x_0 \) of \( K \), there exists a neighborhood \( U \) of \( x_0 \) such that one of the following conditions is satisfied:

\[
\text{(K1)} \quad \text{For } x \in U, \text{ we have } \quad c_1 |x|^\alpha - 1 \leq |\nabla K(x)| \leq c_2 |x|^\alpha - 1
\]

for some constant \( \alpha \geq n - 2 \).

\[
\text{(K2)} \quad \text{For } x \in U \text{ we have } \quad |\nabla^k K(x)| \leq c |\nabla K(x)|^{\frac{\alpha - k}{\alpha - 1}},
\]

where \( 2 \leq k \leq \alpha = n - 2 \).

**Theorem 2.2.** Assume that (i) \( K \in C^1 \) for \( n = 3 \), (ii) for \( n \geq 4 \), at any critical point of \( K \), either (K1) or (K2) is satisfied. Suppose that \( u \) is a positive solution of

\[
\Delta u + K(x) u^{\frac{n+2}{n-2}} = 0 \quad \text{in } B_1. \tag{11}
\]

Then for any \( r \in (0, \frac{1}{2}) \), we have

\[
\left( \max_{B_r} u \right) \left( \min_{B_{2r}} u \right) \leq c r^{2-n}. \tag{12}
\]

Furthermore, if \( u_i \) is a sequence of solutions of (12), then any blow up point is a simple blow up point.

When \( u \) is a global solution defined on \( S^n \), then Theorem 2.2 was proved by Chang-Gursky-Yang for \( n = 3 \), Schoen-Zheng, for \( n = 3, 4 \) and Y.Y. Li for \( n \geq 4 \). In [CLn2], the authors proved Theorem 2.2 via the method of moving planes. For the details, we refer the reader to [CLn2]. An immediate consequence of Theorem 2.2 is that any blow up point must be a critical point of \( K \).
3. Main Theorems

In this section, we always assume \( K \in C^1(\mathbb{B}_1) \) and satisfies the following conditions:

(K3) For any \( \varepsilon > 0 \), there exists \( c(\varepsilon) > 0 \) such that \( c(\varepsilon) \leq |\nabla K(x)| \leq c_1 \) for \( |x| \geq \varepsilon \) where \( c_1 \) is a positive constant independent of \( i \) and \( \varepsilon \).

(K4) The origin is a critical point of \( K \) and \( K(x) = K(0) + Q(x) + R(x) \) in a neighborhood of 0 where \( Q(x) \) is a \( C^1 \) homogeneous function of order \( \alpha > 1 \) satisfying
\[
\begin{align*}
c_1|x|^{\alpha-1} \leq |\nabla Q(x)| \leq c_2|x|^{\alpha-1},
\end{align*}
\]
and both \( R(x)|x|^{-\alpha} \) and \( |\nabla R(x)||x|^{1-\alpha} \) tend to zero as \( |x| \to 0 \).

Let \( U_0 \) be the positive solution of
\[
\Delta U_0 + K(0)U_0^{n+2} = 0 \quad \text{in} \quad \mathbb{R}^n.
\]
(13)

Then, \( Q \) in (K4) satisfies
\[
[Q] \quad \left( \begin{array}{c}
\int_{\mathbb{R}^n} \nabla Q(\xi + y)U_0^{2n-2}(y)dy \\
\int_{\mathbb{R}^n} Q(\xi + y)U_0^{2n-2}(y)dy
\end{array} \right) \neq \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \quad \text{for all} \quad \xi \in \mathbb{R}^n.
\]

The first result in this section is

**Theorem 3.1.** Suppose \( \{u_i\} \) is a sequence of positive solutions of (11). Assume (K3) and (K4) with \( 1 < \alpha < n - 2 \). If \( Q \) satisfies
\[
\int_{\mathbb{R}^n} Q(\xi + y)U_0^{2n-2}(y)dy > 0
\]
whenever \( \int_{\mathbb{R}^n} \nabla Q(\xi + y)U_0^{2n-2}(y)dy = 0 \). Then \( u_i \) is uniformly bounded in \( \mathbb{B}_{1/2} \).

**Remark 3.2** If \( \alpha \geq n - 2 \), then Theorem 3.1 does not hold in general. For a counter example, please see [LL].

**Theorem 3.3.** Assume (K3) and (K4) hold. Suppose 0 is a blow-up point of a sequence of solutions of (11). Then 0 is an isolated blow up point. Furthermore, the inequality
\[
u_i(x)|x|^{n-2} \leq C\]
(14)
holds for $|x| \leq \frac{1}{2}$.

Let $u_i(x_i) = \max_{B_{\frac{1}{2}}} u_i$. Then, by (14), we have

$$\xi = \lim_{i \to +\infty} M_i^{\frac{n}{n-2}} x_i,$$

In the proof of Theorem 3.3, $\xi$ satisfies

$$\int_{\mathbb{R}^n} \nabla Q(\xi + y) U_0^{\frac{2n}{n-2}}(y) dy = 0,$$  \hspace{1cm} (16)

$$\int_{\mathbb{R}^n} \nabla Q(\xi + y) U_0^{\frac{2n}{n-2}}(y) dy \leq 0,$$  \hspace{1cm} (17)

where $U_0$ is the solution of (13).

By assuming $[Q]$, we have more precise description of $u_i(x)$ near its blow up point.

**Theorem 3.4.** Suppose $(K3)$, $(K4)$ and $[Q]$ with $\frac{n-2}{2} \leq \alpha < n-2$ are satisfied. Assume $0$ is a blow up point of a sequence of solutions $u_i$. Let $M_i = \max_{B_{\frac{1}{2}}} u_i$, and $m_i = \min_{B_{\frac{1}{2}}} u_i$. Then there exists a constant $c > 0$ such that

$$u_i(x + x_i) \leq cM_i^{-\beta} |x|^{2-n} \text{ for } |x| \leq M_i^{-\beta},$$ \hspace{1cm} (18)

where $\beta = \frac{2}{n-2} (1 - \frac{\alpha}{n-2})$.

$$u_i(x + x_i) \sim M_i^{1-\frac{2\alpha}{n-2}} \text{ for } |x| \geq M_i^{-\beta}.$$ \hspace{1cm} (19)

In particular,

$$\begin{cases} 
\lim_{i \to +\infty} m_i = 0 & \text{if } \alpha > \frac{n-2}{2}, \\
m_i \sim 1 & \text{if } \alpha = \frac{n-2}{2}.
\end{cases}$$

Furthermore, we have

$$\lim_{i \to +\infty} \int_{B_1} K(x) u_i^{\frac{2n}{n-2}} dx = S_n^{\frac{n}{2}} \text{ if } \alpha > \frac{n-2}{2},$$

and

$$\lim_{i \to +\infty} \int_{B_r} K(x) u_i^{\frac{2n}{n-2}} dx = S_n^{\frac{n}{2}} (1 + o(1)) \text{ if } \alpha = \frac{n-2}{2},$$
where $K(0) = n(n-2)$ is assumed.

For $\alpha < \frac{n-2}{2}$, we have

**Theorem 3.5.** Suppose the assumption of Theorem 3.4 holds except that $\alpha$ satisfies $1 < \alpha < \frac{n-2}{2}$. Then

$$\lim_{i \to +\infty} \int_{B_{\frac{1}{i}}} u_i^{2n-(n-2)}(x) dx = +\infty.$$  

Furthermore, there exists a subsequence of $u_i$ (still denoted by $u_i$) such that $u_i$ converges to a singular solution $u$ of (11) with $0$ as a nonremovable singularity. The conformal metric $ds^2 = u^{-\frac{n-2}{2}}|dx|^2$ is complete in $B_{\frac{1}{i}} \setminus \{0\}$. If we assume $0$ is the only zero of

$$\int_{\mathbb{R}^n} \nabla Q(\xi + y)U_0^{\frac{2n}{n-2}}(y) dy = 0.$$  

Then $u(x) = \bar{u}(|x|)(1 + o(1))$ as $|x| \to 0$.

For the proofs of Theorem 3.1 ~ 3.5, we refer the reader to [CLn3]. As an application, we have

**Theorem 3.6.** Let $K(x)$ be a Morse function on $S^5$, and satisfy $\Delta K(P) \neq 0$ for any critical point $P$ of $K$. Then there exists a constant $C > 0$ such that for any conformal metric $g = u^4 g_0$ with $K(x)$ as the scalar curvature, we have

$$C^{-1} \leq u(x) \leq C \text{ for } x \in S^5,$$

Let $d$ denote the Leray-Schauder degree among all solutions. Then $d = 0$.

**REFERENCES**


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