AN APPROXIMATION THEOREM FOR ORDER BOUNDED OPERATORS

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The object of this paper is to outline some recent work with P.G. Dodds, B. de Pagter and A.R. Schep [1]. In the following $E$ and $F$ will be Riesz spaces and $T$ will be a positive operator from $E$ to $F$. For proofs which are not given the reader is referred to a forthcoming paper [1]. Our aim is to approximate in a purely order theoretic way any operator in the order interval $[0,T]$ of the space of all regular operators between $E$ and $F$ with operators of a particularly simple kind with respect to $T$. For the sake of convenience we will assume that $E = C(K)$ (except in corollary 7), that the normal integrals on $F$, denoted $F^-$, separate the points of $F$ and that $F$ is Dedekind complete. The latter has as a consequence that the space of all order bounded (= regular) operators from $E$ to $F$, denoted by $L_b(E,F)$ is itself a Dedekind complete Riesz space.

Every element $f \in C(K)$ determines a multiplication operator $g \to gf$ on $C(K)$, which is called a multiplier. Abstractly such operators $\sigma : C(K) \to C(K)$ are defined by the conditions that $|\sigma(g)| \wedge |h| = 0$ whenever $|g| \wedge |h| = 0$ and that $\sigma$ is order bounded.

We are interested in the set of all operators $R$ in $[0,T]$ for which there exist $n \in \mathbb{N}$, multipliers $\sigma_1, \ldots, \sigma_n$ on $C(K)$ and order projections $\pi_1, \ldots, \pi_n$ on $F$ such that $R = \sum_{i=1}^n \pi_i T \sigma_i$. The set of all those operators will be labelled $\mathcal{L}(T)$. The elements of $\mathcal{L}(T)$ serve as approximating operators in $[0,T]$.

The following terminology is needed. If $L$ is a Riesz space and
φ is an order bounded functional on L, then \( \rho_\varphi(f) = |\varphi(|f|)| (f \in L) \) defines a seminorm on L. For a set of order bounded functionals \( M \) on L, we define \( |\sigma|(L,M) \) to be the locally convex topology generated by the set of all seminorms \( \rho_\varphi \) with \( \varphi \in M \). Every \( 0 \leq \varphi \in F_n^{-} \) and every \( 0 \leq x \in E \) determines an element \( \phi_{\varphi,x} \) in the space of normal integrals on \( L_b(E,F) \) by \( \phi_{\varphi,x}(S) = \langle SX, \varphi \rangle \) for all \( S \in L_b(E,F) \).

Taking \( F = \{ \phi_{\varphi,x} | 0 \leq \varphi \in F_n^{-}, 0 \leq x \in E \} \), we have all the notation to state lemma 1.

**Lemma 1.** \( \ell(T) \) is \( |\sigma|(L_b(E,F),F)\)-dense in \([0,T]\).

The proof of lemma 1 is largely based on a convenient formula for the infimum of two positive operators from E to F. Indeed, for every \( 0 \leq S, R \in L_b(E,F), 0 \leq x \in E \) and \( 0 \leq \varphi \in F_n^{-} \) we have \( \langle (R \wedge S)(x), \varphi \rangle = \inf \sum \langle \pi_j \pi_i, \varphi \rangle \wedge \langle \pi_j R \pi_i, \varphi \rangle \), where the infimum is taken over all finite subsets \( \{ x_1, ..., x_n \} \subset E^+ \) with \( \sum_i x_i = x \) and all finite subsets of mutually disjoint band projections \( \{ \pi_1, ..., \pi_m \} \) on F with \( \sum_j \pi_j = Id_F \).

However, we have in mind a more intrinsic way of characterizing \([0,T]\) in terms of \( \ell(T) \). For this purpose we need more structural information about \( \ell(T) \). By considering the tensor product of the band projections on F with the multipliers on E we obtain the following result.

**Lemma 2.** \( \ell(T) \) is a sublattice of \([0,T]\).

If L is a Riesz space and \( K \) is a subset of L, \( IK \) is defined to be the set of all \( f \in L \) for which there exists a subset \( \{ f_\tau \} \subset K \) with \( f_\tau \uparrow f \). \( DK \) is defined by replacing \( \uparrow \) in the preceding sentence by \( \downarrow \) (and \( IK \) by \( DK \)). The following up-down theorem by D.H. Fremlin suits the situation (see [3]).
Theorem 3. If $L$ is a Dedekind complete Riesz space, if $M$ is a solid subspace of the normal integrals on $L$ which separates the points of $L$ and if $K$ is a sublattice of $L$, then the closure of $K$ for $|\sigma(L,M)$ is $\text{VIDIK}$.

We employ theorem 3 by taking $L = L_b(E,F)$, $M = F$, $K = \ell(T)$. Lemma 1, lemma 2 and some routine inspections of the situation, together with theorem 3 now yield:

Theorem 4. $\text{VIDI} \ell(T) = [0,T]$.

Because the characterization in theorem 4 is intrinsic, we can now derive a much stronger approximation theorem, (due to Kalton and Saab [4]).

Theorem 5. If $\rho$ is an order continuous Riesz seminorm on the principal ideal generated by $T$ in $L_b(E,F)$, if $S \in [0,T]$ and $\epsilon > 0$, then there exists $S' \in \ell(T)$ with $\rho(S - S') < \epsilon$.

Apart from being interesting in their own right, these theorems have nice consequences. The main reason for this is the preservation of certain properties of $T$ in $\ell(T)$. For instance, every element of $\ell(T)$ is compact if $T$ is compact. A straightforward application is the following majorization result by Dodds and Fremlin (see [2]).

Corollary 6. If $F$ is an AL-space and $E = C(K)$, if $0 \leq S \leq T$ are operators from $E$ to $F$, and $T$ is a compact operator, then $S$ is a compact operator.

To discuss another application we have to abandon the assumption $E = C(K)$. Instead, we assume that $E$ is a Banach lattice with quasi-interior point, i.e. with an element $u \in E$ such that $E$ is norm dense
in the principal ideal generated by \( u \). We borrow the abstract definition for the multipliers from the \( C(K) \) situation, i.e. the multipliers are the order bounded operators \( \sigma : E \to E \) with \( |\sigma(g)| \wedge |h| = 0 \) as soon as \( |g| \wedge |h| = 0 \). The multipliers form a Riesz space under pointwise operations and, in fact, this Riesz space is Riesz isomorphic to a \( C(K) \)-space. Using the same techniques, the statements in theorem 4 and theorem 5 remain valid. The latter will be used in the proof of our next corollary. (Again due to Kalton and Saab [4]).

**Corollary 7.** If \( E \) and \( F \) are Banach lattices and \( F \) has order continuous norm (so no restrictions on \( E \) at all), if \( 0 \leq S \leq T \) are operators from \( E \) to \( F \) and \( T \) is a Dunford-Pettis operator, then \( S \) is a Dunford-Pettis operator.

We sketch a proof of this corollary. To prove that \( S \) is a Dunford-Pettis operator we have to show that for every sequence \( (a_n)_{n \in \mathbb{N}} \) of elements of \( E \), which converges weakly to zero, \( \|S(a_n)\| \to 0 \). Suppose \( a_n \to 0 \) weakly. By taking \( y = \sum_{n=0}^{\infty} 2^{-n} |a_n| \) we may assume that \( E \) has a quasi-interior point, namely \( y \). Let \( A \) be the solid hull of \( \{a_n \mid n \in \mathbb{N}\} \) and \( B = \{\varphi \in F^* \mid \|\varphi\| \leq 1\} \). Define for every \( R \) in the principal ideal generated by \( T \) in \( L_b(E,F) \), \( \rho(R) = \sup\{|(Ra,\varphi)| \mid a \in A, \varphi \in B\} \). It can be shown that \( \rho \) is an order continuous Riesz seminorm on the principal ideal generated by \( T \) in \( L_b(E,F) \). Therefore, there exists by the remarks preceding corollary 7 an element \( S' \) in \( \ell(T) \) with \( \rho(S - S') \leq \varepsilon \). As \( S' \) is a Dunford-Pettis operator it easily follows that \( \|Sa_n\| \to 0 \).

References

theorems in the centre of $L_b(E,F)$.


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