SOME THEOREMS ON ORLICZ-SOBOLEV SPACES,
AND AN APPLICATION TO NEMITSKY OPERATORS

Grahame Hardy

1. INTRODUCTION

We are concerned here with the problem of extending, to Orlicz-Sobolev
spaces, certain theorems of Marcus and Mizel on Nemitsky operators on
Sobolev spaces. (See [5].)

Marcus and Mizel's proofs rely upon, in particular,

(i) Gagliardo's characterisation of the Sobolev space \( W_{1,p} \) in
terms of absolute continuity; and

(ii) bounds and limits of difference quotients in Sobolev spaces.

We shall give suitable extensions of (i) and (ii) to
Orlicz-Sobolev
spaces in §§ (2) and (3) below, which enables us to give an extension
of the theorems of Marcus and Mizel. (See § 4.)

2. ORLICZ-SOBOLEV SPACES AND THE SPACES \( A(\Omega) \)

Throughout this paper, \( \Omega \) denotes a domain in \( \mathbb{R}^n \).

Since all the definitions of both Orlicz and Orlicz-Sobolev spaces
which occur in the statements of our theorems can be found in [1], we
shall not repeat them here. For the spaces \( A(\Omega) \) (i.e., Beppo Levi
spaces), we shall follow [5]. (With a minor difference in notation,
essentially that, in denoting certain equivalence classes, we use "-" instead of a dash, to avoid an obvious source of confusion.)
Thus \( \mathcal{A}(\Omega) \) denotes the class of real measurable functions \( u \) on \( \Omega \) such that, for almost every line \( \tau \) parallel to any co-ordinate axis, \( u \) is locally absolutely continuous on \( \tau \cap \Omega \). \( \mathcal{A}(\Omega) \) denotes the class of functions \( u \) such that \( u \) coincides almost everywhere in \( \Omega \) with a function \( \tilde{u} \) in \( \mathcal{A}(\Omega) \). For \( u \in \mathcal{A}(\Omega) \), \( \tilde{D}_j u \) (or \( \tilde{D}_x^j u \)), the strong approximate derivative of \( u \) with respect to \( x_j \), denotes any member of the equivalence class of functions measurable on \( \Omega \) which contains the classical partial derivative \( D_j u \). We shall use \( \partial_j u \) or \( \partial_x^j u \) to denote a weak derivative. Our extension of Gagliardo's theorem is:

**THEOREM 1**  
Let \( M \) be an \( N \)-function, and suppose \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with the cone property. Then a function \( u \) defined on \( \Omega \) belongs to \( W^1 L_M(\Omega) \) if and only if

(a) \( u \in \mathcal{A}(\Omega) \);

(b) \( \tilde{D}_j u \in L_M(\Omega), \quad j = 1, \ldots, n. \)

Moreover, if \( u \in W^1 L_M(\Omega) \), then \( \tilde{D}_j u = \partial_j u \) almost everywhere in \( \Omega \).

Using Theorem 1 (instead of Gagliardo's Theorem), we obtain the following version of a chain rule due to Serrin.

**THEOREM 2**  
Let \( f: \mathbb{R} \to \mathbb{R} \) be locally absolutely continuous, let \( M \) be an \( N \)-function and suppose \( u \in W^{1,1}_1(\Omega) \). Then \( f \circ u \in W^1 L_M(\Omega) \) if and only if

(i) \( (f \circ u) \partial_j u \in L_M(\Omega), \quad j = 1, \ldots, n, \)

where we make the following convention:

(*) the product is zero if the term on the right is zero.
Moreover, if (i) holds,

\[ \partial_j (f \circ u) = (f' \circ u) \partial_j u, \quad j = 1, \ldots, n, \text{almost everywhere in } \Omega. \]

3. DIFFERENCE QUOTIENTS IN ORLICZ-SOBOLEV SPACES

Definition. For \( u : \Omega \to \mathbb{R}, e_j, 1 \leq j \leq n \), the standard basis for \( \mathbb{R}^n \), and \( x \in \mathbb{R}^n \), we define the difference quotient in the direction \( e_j \) by

\[ \delta^j_h u(x) = \frac{u(x+he_j) - u(x)}{h}, \quad h \neq 0, \text{ whenever } x \text{ and } x + he_j \in \Omega. \]

Using arguments similar to those used to establish the analogous results for Sobolev spaces, (see [2]), we can prove the following:

**THEOREM 3.** Suppose \( \Omega \) is a bounded, and that \( \Omega' \) is an open set such that \( \Omega' \subset \subset \Omega \). Then if \( 0 < |h| < \text{dist}(\Omega', \text{bdry } \Omega) \), and if \( u \in W^{m+1}_M(\Omega) \) for some \( m \geq 1 \),

\[ \| \delta^j_h u \|_{m-1, M, \Omega'} \leq \| u \|_{m, M, \Omega}. \]

Further, if there exists a number \( C \) such that \( \| \delta^j_h u \|_{m, M, \Omega'} \leq C, \quad 1 \leq j \leq n, \)

for every open \( \Omega' \subset \subset \Omega \) and all \( h \) sufficiently small, then \( u \in W^{m+1}_L(M, \Omega) \) and \( \| \delta^j u \|_{m, M, \Omega} \leq C, \quad 1 \leq j \leq n \).

4. NEMITSKY OPERATORS

Definition. A function \( g : \Omega \times \mathbb{R}^m \to \mathbb{R} \) is said to be a generalised locally absolutely continuous (briefly g.l.a.c.) Caratheodory function if:

(i) There exists a null subset \( N \) of \( \Omega \) such that for every fixed \( x \in \Omega \setminus N \) we have
203

(a) \( g(x, \cdot) \) is separately continuous in \( \mathbb{R}^m \);

(b) for every line \( \tau \) parallel to one of the axes in \( \mathbb{R}^m \),
\[ g(x, \cdot) \big|_\tau \text{ is locally absolutely continuous.} \]

(ii) For each fixed \( t \in \mathbb{R}^m \), \( g(\cdot, t) \in \tilde{A}(\Omega) \).

The Nemitsky operator \( G \) is then defined on functions \( u : \Omega \to \mathbb{R}^m \)
by \( (Gu)(x) = g(x, u(x)) \).

Our extension of Marcus and Mizel's theorem (including a corollary)
is then:

THEOREM 4. \( \) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) having the cone property,
and let \( g \) be a g.l.a.c. Caratheodory function on \( \Omega \times \mathbb{R}^m \). Let \( P, Q_k \) and \( Q_k^+, k = 1, \ldots, m \), be \( N \)-functions having the following properties:

(i) \( P \) and \( Q_k, k = 1, \ldots, m \), satisfy the \( \Delta_2 \) condition;

(ii) \( P \leq Q_k, k = 1, \ldots, m \);

(iii) there exist complementary \( N \)-functions \( R_k \) and \( \tilde{R}_k \) such that the inequalities
\[ R_k(s) \leq P^{-1}[Q_k(a_k s)] \]
and
\[ \tilde{R}_k(s) \leq P^{-1}[Q_k^+(\beta_k s)] \]
are satisfied for \( s \geq \alpha_k \), where \( a_k, \beta_k, \alpha_k, \beta_k, \gamma_k, k = 1, \ldots, m \), are constants.

Suppose \( a, b, a_k, \beta_k, \gamma_k \) are functions such that for every fixed \( t \in \mathbb{R}^m \)

(iv) \[ |\tilde{D}_{x_i} g(x, t)| \leq a(x) + b(t) \text{ a.e. in } \Omega, i = 1, \ldots, n; \text{ and} \]
the inequality
holds at every point \((x,t) \in (\Omega \setminus N) \times \mathbb{R}^m\) at which the derivative exists in the classical sense. (Here \(N\) is the null set of the definition above.)

Furthermore, \(a, b, a_k\) and \(b_{k,j}\) have the properties (vi) - (x) listed below:

(vi) \(0 \leq a \in L^p(\Omega)\);

(vii) \(b\) is non-negative and separately continuous in \(\mathbb{R}^m\);

(viii) \(0 \leq a_k \in L^q_k(\Omega), \ k = 1, \ldots, m;\)

(ix) \(0 \leq b_{k,j} \) is an extended real valued Borel function on \(\mathbb{R}, \ k, j = 1, \ldots, m;\)

(x) \(b_{k,k} \in L^1_{\text{loc}}(\mathbb{R}), \ k = 1, \ldots, m.\)

Let \(u_k \in W^{1,p}_k(\Omega), \ k = 1, \ldots, m, \) let \(u = (u_1, \ldots, u_m)\), and suppose that

(xii) \(bou \in L^p(\Omega)\);

(xiii) \(b_{k,j}ou_k \in L^p_{\text{loc}}(\Omega)\) \(k, j = 1, \ldots, m, k \neq j;\)

and, with the convention (*),

(xiii) \([b_{k,k}ou_k] \partial_i u_k \in L^p(\Omega), \ k = 1, \ldots, m, i = 1, \ldots, n.\)

Then \(Gu\) belongs to \(W^{1,p}(\Omega)\), and, with the convention (*),

\[
|\partial_i (Gu(x))| \leq a(x) + (bou)(x) + \\
+ \sum_{k=1}^m \left[ a_k(x) + \sum_{j=1}^m (b_{k,j}ou_j)(x) |\partial_i u_k(x)| \right],
\]

almost everywhere in \(\Omega, i = 1, \ldots, n.\)

NOTE. Families of N-functions satisfying (i), (ii), and (iii) can be constructed from standard N-functions (such as those listed in [4]), using the following:
Proposition. Let $P$ and $R$ be $N$-functions satisfying the $\Delta_2$ condition, and let $Q = P \circ R$, $Q^r = P \circ \tilde{R}$. Then $Q$ and $Q^r$ are $N$-functions having the properties:

(i) $Q$ satisfies the $\Delta_2$ condition;
(ii) $P < Q$;
(iii) $R = P^{-1} \circ Q$, $\tilde{R} = P^{-1} \circ Q^r$.

REFERENCES


School of External Studies
University of Queensland
St Lucia QLD 4067
AUSTRALIA