A Banach space [dual space] $X$ satisfies the weak [weak*] Opial condition if whenever $(x_n)$ converges weakly [weak*] to $x_\infty$ and $x_0 \neq x_\infty$ we have

$$\liminf_n \|x_n - x_\infty\| < \liminf_n \|x_n - x_0\|.$$ 

Zdzisław Opial [1967] introduced the weak condition to expand upon results of Browder and Petryshyn [1966] concerning the weak convergence of iterates for a nonexpansive selfmapping of a closed convex subset to a fixed point. In particular he observed that a uniformly convex Banach space with a weak to weak* sequentially continuous support mapping satisfies the weak condition. A support mapping is a selector for the duality map

$$D: X \to 2^{X^*}; \ x \mapsto \{f \in X^*: f(x) = \|f\|^2 = \|x\|^2\}.$$ 

Uniform convexity is not sufficient for the weak to weak* sequential continuity of the unique support mapping. Browder [1966], and independently Hayes and Sims in connection with operator numerical ranges, had observed that the uniformly convex space $L_1[0, 1]$ does not have a weak to weak (= weak*) continuous support mapping, while all of the sequence spaces $\ell_p$ $(1 < p < \infty)$ do. Opial [1967] demonstrated that with the exception of $p = 2$ none of the spaces $L_p[0, 1]$ have weak to weak continuous support mappings. Indeed, Fixman and Rao characterize $L_p(\Omega, \Sigma, \mu)$ spaces with a weak to weak continuous support mapping as those spaces for which every element of $\Sigma$ with finite positive measure contains an atom.

That uniform convexity is not necessary is shown by the example of $\ell_1$ with an equivalent smooth dual norm. That the unique support mapping is
weak to weak* sequentially continuous follows from the norm to weak* upper semi-continuity of a duality mapping and the fact that \( \ell_1 \) is a Schur space.

These early results were considerably improved by Gossez and Lami Dozo [1972]. In particular they show the following.

1. The assumption of uniform convexity is unnecessary for Opial's result: Any Banach space [dual space] with a weak [weak*] to weak* sequentially continuous support mapping satisfies the weak [weak*] Opial condition.

   Indeed, their proof is easily adapted to show that a space has the weak [weak*] Opial condition if the Duality mapping is such that given any weak* - neighbourhood \( N \) of zero, if \( (x_n) \) converges weakly [weak*] to \( x_\infty \) then eventually \( D(x_n) \cap (D(x_\infty)^+) \neq \emptyset \).

2. The weak Opial condition implies the fixed point property for non-expansive self-maps of weak-compact convex sets. We give a direct proof [Van Dulst, 1982] which also applies in the weak* case.

Proposition 1: Let \( x \) be a Banach space [dual space] with a weak* - sequentially compact ball\(^1\) satisfying the weak [weak*] Opial condition. If \( C \) is a weak [weak*] - compact convex subset of \( x \), then any non-expansive mapping \( T: C \to C \) has a fixed point.

Proof: Choose \( x_0 \in C \), then since \( C \) is closed and convex, for any \( n \) the mapping \( (1 - \frac{1}{n})T + \frac{1}{n}x_0 \) is a strict contraction on \( C \) which by the Banach contraction mapping principle has a unique fixed point \( x_n \) in \( C \).

Using the boundedness of \( C \) it follows that

\[
\|x_n - Tx_n\| \to 0.
\]

Passing to a subsequence if necessary we may also assume that \( (x_n) \) converges weak [weak*] to a point \( x_\infty \).

\(^{1}\)For example; the dual of a separable space, or more generally the dual of any smoothable space.
Then,

$$\liminf_n \|Tx_n - x_n\| = \liminf_n \|Tx_n - x\|$$

$$\leq \liminf_n \|x_n - x\|$$

contradicting the weak [weak*] Opial condition unless $Tx_n = x_n$.

Gossez and Lami Dozo [1972] in fact proved that the weak Opial condition implies normal structure thereby deducing the weak version of the above result via Kirk [1965].

Whether or not the weak* Opial condition implies normal structure for weak* compact convex sets remains an open question.

(3) Weak to weak* sequential continuity of a support mapping is not necessary for the weak Opial condition. For $1 < p < q < \infty$ the space $(\ell_p \oplus \ell_q)_2$ satisfies the weak Opial condition, but [Bruck, 1969] the unique support mapping is not weak to weak continuous.

Karlovitz [1976] explored other connections between the Opial conditions and the space's geometry, establishing a relationship with approximate symmetry in the Birkhoff-James notion of orthogonality.

The purpose of this note is to provide the following characterization of the weak [weak*] Opial condition in terms of support mappings.

**Theorem 2:** The Banach space [dual space] $X$ satisfies the weak [weak*] Opial condition if and only if whenever $(x_n)$ converges weakly [weak*] to a non-zero limit $x_\infty$ there exists a $\delta > 0$ such that eventually $D(x_n)x_\infty \subset [\delta, \infty)$.

**Proof:** (\Rightarrow) Assume this were not the case, then by passing to subsequences we can find $(x_n)$ converging weakly [weak*] to $x_\infty$ with $\|x_n\| \geq \|x_\infty\| > 0$ and $f_n \in D(x_n)$ such that $\lim_{n} f_n(x_\infty) \leq 0$. 

\[
But
\[
\liminf_{n} \|x_n\|^2 = \liminf_{n} \|x_n - 0\|^2 \\
> \liminf_{n} \|x_n - x_\infty\|^2 \\
\geq \liminf_{n} f_n(x_n - x_\infty) \\
= \liminf_{n} (\|x_n\|^2 - f_n(x_\infty)) \\
= \liminf_{n} (\|x_n\|^2 - \lim f_n(x_\infty)),
\]
whence \(\lim f_n(x_\infty) > 0\), a contradiction.

(\(\Leftarrow\) a modification of the proof in Gossez and Lami Dozo [1972].)

Using the integral representation for the convex function
\(t \mapsto \frac{1}{2}\|x + ty\|^2\) [Roberts and Varberg, 1973, 12 Theorem A] we have
\[
\frac{1}{2}\|x + y\|^2 = \frac{1}{2}\|x\|^2 + \int_{0}^{1} g^+(x + ty; y) \, dt
\]
where
\[
g^+(u; y) = \lim_{h \to 0^+} \frac{\frac{1}{2}\|u + hy\|^2 - \frac{1}{2}\|u\|^2}{h}
\]

To establish the weak [weak*] Opial condition it suffices to show that if \(y_n\) converges weakly [weak*] to \(y_\infty \neq 0\) then
\[
\liminf_{n} \frac{1}{2}\|y_n\|^2 > \liminf_{n} \frac{1}{2}\|y_n - y_\infty\|^2.
\]

Now,
\[
\frac{1}{2}\|y_n\|^2 = \frac{1}{2}\|y_n - y_\infty\|^2 + \int_{0}^{1} g^+(y_n - y_\infty + ty_\infty; y_\infty) \, dt
\]
So
\[
\liminf_{n} \frac{1}{2}\|y_n\|^2 \geq \liminf_{n} \frac{1}{2}\|y_n - y_\infty\|^2 \\
+ \liminf_{n} \int_{0}^{1} g^+(y_n - y_\infty + ty_\infty; y_\infty) \, dt.
\]

By Fatou's lemma [Halmos, 1950] it is therefore sufficient to prove for each \(t \in (0, 1)\) that
\[
\liminf_{n} g^+(y_n - y_\infty + ty_\infty; y_\infty) > 0.
\]
But, 

\[ g^+(y_n - y_\infty + ty_\infty; y_\infty) = \operatorname{Max}\{f(y_\infty); f \in D(y_n - y_\infty + ty_\infty)\} \]

[Barbu and Precupanu, 1978, §2.1 example 2° and Proposition 2.3] and

\( y_n - y_\infty + ty_\infty \) converges weakly [weak\*] to \( ty_\infty \neq 0 \), so for \( n \) sufficiently large and some \( \delta > 0 \) we have \( f(ty_\infty) > \delta \) for all \( f \in D(y_n - y_\infty + ty_\infty) \)

\[ \square \]

Remarks:

(1) Using the weak* - neighbourhood \( \{g \in X^*; g(x_\infty) > -\frac{1}{2}\|x_\infty\|^2\} \) of 0 in \( X^* \) it is easily seen that the condition of the theorem is satisfied if the Duality mapping is sequentially weak [weak*] to weak* upper semi-continuous.

(2) From the details of the proof we see that if for some selection of \( f_n \) from \( D(x_n) \) we have \( \liminf_n f_n(x_\infty) > 0 \), where \( x_n \) converges weak [weak*] to \( x_\infty \neq 0 \), then the same is true for all selections.

REFERENCES


University of New England
Armidale N.S.W. 2351
AUSTRALIA.